A
NEW KEY
TO THE
EXACT SCIENCES:
OR,
A NEW AND PRACTICAL THEORY

BY WHICH MATHEMATICAL PROBLEMS OR ALGEBRAIC
EQUATIONS OF ALMOST EVERY DESCRIPTION
CAN BE SOLVED WITH ACCURACY,

AND WITH GREATER FACILITY AND SIMPLICITY THAN THEY CAN
BE BY ANY METHOD THAT HAS YET BEEN GIVEN
BY ANY OTHER AUTHOR:

In which are also introduced,

A VARIETY OF

Useful and Interesting Problems,

THAT HAVE NEVER BEFORE BEEN PROPOSED,

And which it is believed cannot be solved by any methods or rules except those here laid down.

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PREFACE.

SO much has already been said by different authors, by way of preface, about the necessity, the utility, and the advantages that are derived from cultivating the mathematical sciences, that it would be difficult to add any thing on that subject, without repeating what has been often stated by others; but as the rules and solutions given in the present work, will differ materially from those which have been given by former authors on the same subject, some further remarks are in a measure necessary.

Algebra, the subject of the present treatise, is certainly one of the principal branches of the mathematical sciences, and has frequently been called the master-key to all the rest; but may, perhaps, with equal propriety be called the calculation or arithmetic of nature; and is supposed to have been much better understood by the ancients than at present. Indeed, the art appears to have been in a great measure lost to the Europeans for a considerable length of time, and to have arrived at its present imperfect state by very slow degrees; and I

am induced to believe the principal reason why a science so much to be valued has not been more generally cultivated and understood, arises chiefly from the laborious, mysterious, and I think I may add unnecessary investigations in which the rules for the solutions are involved. Where propositions, simple in themselves, are rendered abstruse by the fine spun, logical reasoning, used to elucidate, explain or resolve them, it frequently tends more to perplex and distract the mind, and lead to error, than to elicit truth. Another equally or perhaps more injurious tendency is, that it often deters youth, even of the brightest natural parts, from engaging or continuing to persevere in studies of this nature. I am not singular in expressing my opinion, when I observe, it would appear that nearly all the gentlemen who have written on this science have been more desirous to exhibit their own ingenuity and erudition to those who were in some measure familiar with the science, than to instruct the mere tyro.

From these considerations, I have undertaken the present performance, the intention of which is to enlarge or enrich, and also to simplify, the exact sciences in general, by giving certain spe-

cific rules, founded on a few self-evident principles; but which, as before observed, differ, with the exception of one or two, entirely from those which have been usually given—requiring, in most instances, not one-tenth part of the figures, characters or symbols; on which account, the present work will be found to contain nearly twice as much matter as is contained in the same space in any other work on the same subject. It has been my particular wish to give the rules with so much plainness, that they may be fully as easy to be understood by a tyro, as any of the rules in common arithmetic: and it can be asserted, on the confidence of actual experience, that in twelve or fifteen days, a youth, acquainted with vulgar and decimal fractions, and the extraction of the roots, will, with the assistance of a competent teacher, acquainted with the principles and practical application of these rules, acquire a knowledge of resolving simple and quadratic equations, which frequently is not acquired by six months' close application.

My first intention was to publish a complete system of practical algebra, including simple, quadratic, and cubic equations, with mixed equations, and of

the higher powers. Particular circumstances having rendered that for the present impracticable, and as the change in the system which I propose, applies more generally and forcibly to quadratic, cubic, and the higher powers, than to simple equations, I have been induced to submit the present compendium.—Should this meet with sufficient patronage, a second number, now in forwardness, will be published, giving the application of the rules here laid down, to the solution of every description of equations that has been proposed by the most popular authors, and a number of others that are thought to be original: also, an entirely new method of solving cubic equations and equations of different dimensions; and a third number, will give the application of algebra to geometry. The whole three numbers will contain, it is thought, about two hundred octavo pages, and it is my particular wish to make the whole work a more complete system of practical algebra, than any that has yet been published.

ON THE

SCIENCE OF ALGEBRA.

IN all calculations, whether in what is called Common or Mercantile Arithmetic, or in the branches of Mathematics, the object is to find, from some given quantity or quantities, a certain required unknown quantity. So is algebra applied, in the most general manner, to the relation or comparison of abstract quantities, by certain rules and methods, whereby such questions and propositions are resolved or answered, as will not come within the compass of common arithmetic.

In all algebraic calculations, the numbers or quantities, whether given or required, are not expressed in general by the common method of figures, but by some letter in the alphabet—the given quantities, for the sake of distinction, by the initial letters a, b, c, d, &c., and the unknown quantities by the final letters u, w, x, y, z. But in the present number, it being intended principally to initiate the pupil, all the given quantities will be expressed by figures, and the intermediate and unknown quantities only by letters. The general use of letters may after this be adopted, without that inconvenience which would result from commencing with them too soon.

There are also in algebra certain signs or notes made use of to show the relation and dependance of quantities one upon another; which it is necessary the learner should first of all be well acquainted with.

The sign $+$ *plus*, or more, signifies that the quantity to which it is prefixed is to be added: thus, $6+2$ signifies that 2 is to be added to 6; or $a+b$ signifies that the quantity expressed by b is to be added to the quantity expressed by a.

NOTE.—A quantity which has no sign prefixed, as the leading quantities in each of the above examples, is to be

understood as having the sign $+$ before it; as $+b$ and $+a$, or $+6+2$, and $+a+b$.]

The sign $-$ *minus*, or less, signifies that the quantity which it precedes is to be subtracted: thus, $8-3$ signifies that 3 is to be subtracted from 8; and $a-b$ signifies that the quantity represented by b is to be subtracted from the quantity represented by a. [NOTE.—Those quantities to which the sign $+$ is prefixed, are called *positive*, or *affirmative;* and those to which the sign $-$ is prefixed, are called *negative.*]

The sign $\times$ signifies that the quantities between which it stands are to be multiplied together: thus, 4×6 signifies that 4 is to be multiplied by 6; or $a\times b$ signifies that the quantity represented by a is to be multiplied by the quantity represented by b; also, when any number of letters are joined without any sign between them, it signifies that they are to be multiplied together: thus, ab, abc, $abcd$, signifies that the quantity represented by each of these letters is to be continually multiplied by the other.

The sign $\div$ signifies that the quantity preceding it is to be divided by the quantity that comes after it: thus, $24\div 8$ signifies that the number 24 is to be divided by 8; and $c\div d$ signifies that the quantity represented by c is to be divided by that represented by d. [NOTE.—Division is also sometimes expressed as a vulgar fraction: thus, $\frac{28}{4}$ signifies 28 divided by 4; and $\frac{c}{d}$ signifies that the quantity expressed by c is to be divided by d.]

The sign $=$, called the sign of equality, is used to denote that the quantities standing on each side of it are equal: thus, $7+5=12$ shows that 7 added to 5 is equal to 12; and $x=2m+d$ shows that x is equal to twice the quantity represented by m more the quantity represented by d.]

The notes : :: : signify that the quantities standing between each of them are proportional: thus, as $4:6::12:18$ denotes that 4 is in the same proportion to 6 as 12 is to 18; or thus, as $a:b::c:d$ denotes that a is in the same proportion to b as c is to d.

Numbers that are involved to a square, cube, or any higher power, are expressed by placing the index of the power to which they are involved above the letters or figures: thus, 42^2 signifies the square of 42, or 42 to be involved to the second power; and a^2 signifies the square of a, or a to be involved to the second power: 74^3 signifies the cube of 74; or b^3 signifies the cube or third power of b; and thus are expressed the fourth, fifth, sixth, or higher powers.

The sign $\sqrt{}$ is used to show the root of some power to be extracted. When there is no index, it means the square root. $\sqrt[3]{}$ means the cube root. $\sqrt[4]{}$ means the biquadrate, or the root of the fourth power. $\sqrt[5]{}$ the fifth root, &c.

A *vinculum* is intended to combine two or more numbers, and to express the quantity of those numbers thus combined. It is made different ways; as by a bar, thus, $\overline{b+c}$, signifying the combined number of b added to c; or thus, $\overline{d-e}|^2$, signifying the square of the number formed by subtracting e from d; or thus, by parentheses, $(d+c)^2$, signifying the square of the number formed by adding c to d. This last is the only vinculum used in this work.

There are many other characters or symbols used by different authors, but these, it is believed, are all that are necessary in the present number,

It may not be improper here to give a few practical examples of the notations before given.

1. $6+4-3=7$; that is, 6 more 4, less 3, is equal to 7; or by substituting letters for figures, $a+b-c=d$:
2. *Or*, $7-2+9\times5=70$; that is, 7 less 2, more 9, multiplied by 5, is equal to 70; or by substituting letters, $a-b+c\times d=e$.
3. $6^2+9-5=40$; that is, the square of 6 more 9, less 5, is equal to 40; or by substituting letters, $a^2+b-c=d$.
4. $\sqrt{49}+5\times8=96$; that is, the square root of 49 more 5, multiplied by 8, is equal to 96; or by substituting letters, $\sqrt{b}+a\times c=d$.

OF EQUATIONS IN GENERAL.

THE Doctrine of Equations is that branch of algebra which treats of the different methods of finding the value of some unknown quantity or quantities, by means of the relation they bear to others which are known or given, and may be divided into simple and compound equations.

A *simple equation* is when it contains only the first power of the unknown quantity or quantities.

A *compound equation* is when the unknown quantity or quantities are raised to the second, third, or any higher

power, and are called quadratic, cubic, biquadratic, &c.; though Mr. Bonnycastle and some other authors have given the term of quadratic equations to many that are higher than the second power; and, following their example in the present work, all the equations are called quadratic. Thus, $7+5=12$, is a simple equation, meaning that 7 added to 5 is equal to 12, or $a+b=c$ means that the quantity expressed by a more b (or added to the quantity expressed by b) is equal to the quantity expressed by c. Or, $x^2+5x-6=60$, is a quadratic equation, meaning that the square of x, or of some unknown quantity, added to 5 times that quantity, and 6 subtracted, is equal to the given quantity 60.

OF THE RESOLUTION OF EQUATIONS.

The resolution of simple, and all other equations, means the disengaging the unknown quantity or quantities, with which it or they are connected, and making them stand alone on one side of the equation, so as to be equal to such quantity or quantities, as are given or known on the other side, for the performing of which, several rules of art, or processes founded on certain self-evident principles, are required, each of which will be given, as applicable to the different cases that occur.

I will here give a few of the principal aphorisms, on which the theory now proposed is founded.

I. The product of any two numbers is equal to the square of their mean proportional, less the square of one half their difference.

II. The continued product of any three numbers in arithmetical progression, is equal to the cube of the middle number, less the square of their common difference multiplied by the middle number.

III. The continued product of four numbers in arithmetical progression, is equal to the product of the two means, multiplied by the product of the two extremes.

IV. The sum of the squares of any two numbers, is equal to twice the square of their mean proportional, more twice the square of one half their difference.

V. The sum of the squares of three numbers in arithmetical progression, is equal to three times the square of the middle number, more twice the square of their common difference,

VI. The sum of the squares of four numbers in arithmetical progression, is equal to four times the square of their mean proportional, more five times the square of their common difference.

VII. The sum of the squares of five numbers in arithmetical progression, is equal to five times the square of the middle number, more ten times the square of their common difference.

VIII. The difference of the squares of any two numbers, is equal to their sum multiplied by that difference.

IX. The sum of the cubes of any two numbers, is equal to twice the cube of their mean proportional, more six times the mean proportional multiplied by the square of one half their difference.

X. The sum of the cubes of any three numbers in arithmetical progression, is equal to three times the cube of the middle number, more six times the middle number multiplied by the square of their common difference.

XI. The sum of the cubes of any four numbers in arithmetical progression, is equal to four times the cube of their mean proportional, more fifteen times their mean proportional multiplied by the square of their common difference.

XII. The sum of the cubes of any five numbers in arithmetical progression, is equal to five times the cube of their mean proportional, more thirty times their mean proportional multiplied by the square of their common difference.

XIII. The difference of the cubes of any two numbers, is equal to the square of their mean proportional multiplied by three times their difference, more twice the cube of one half their difference.

These aphorisms may be so modified as to apply to the fourth, fifth or sixth powers, as will appear in the work.

XIV. The continued product of any three numbers in geometrical progression, is equal to the cube of the middle term.

XV. The continued product of four numbers in geometrical progression, is equal to the square of the first term multiplied by the square of the last, or also equal the square of the product of the two means, or of the two extremes.

XVI. The sum of the squares of three numbers in geometrical progression, is equal to three times the square of a mean proportional between the two extremes, more the square of one-half their difference, and the product of the two extremes is equal to the square of the middle term.

XVII. The sum of the squares of four numbers in geometrical progression, is equal to the square of the sum of the two extremes, more the square of the second term multiplied by *the square of the ratio less one.*

XVIII. The continued product of any three numbers in harmonical proportion, is equal to the cube of the second term, more the cube of the difference between the second and third.

XIX. The sum of the squares of any three numbers, in harmonical proportion, is equal to three times the square of the middle number, more three times the square of the difference between the second and third, more the square of the difference between the first and second numbers.

QUADRATIC EQUATIONS.

CASE I.

When the unknown quantity is involved in any number of terms of the square of that quantity, also including a number of simple terms equal to any given quantity, as $3x^2+6x=45=a$; these problems are solved by dividing the whole equation by the number of square terms given: thus,

Given $3x^2+6x=45$, to find x;

by dividing by 3, we have $x^2+2x=15$; and then completing the square, which is by squaring one-half of the simple terms (or, as they are generally termed, the coefficients), and then adding the square thus produced to the given quantity, which will then be a square, (or, as it is sometimes termed, a rational number ;) then extract the root, and add to or sub-

tract from the root thus found the quantity that was squared; i. e. when the sign is + subtract, and when the sign is — add; and the number or quantity thus produced will be $=x$; or the unknown quantity required.

EXAMPLES.

1. Given $x^2+x=72$; to find x.

First, $x\div 2=\frac{1}{2}x$; then $\frac{1^2}{2}=\frac{1}{4}$, and $72+\frac{1}{4}=72\frac{1}{4}$; then $\sqrt{72\frac{1}{4}}=8\frac{1}{2}=x+\frac{1}{2}$; then $8\frac{1}{2}-\frac{1}{2}=8=x$.

$8^2=64+8=72.$ } Proof.

2. Given $x^2+8x=84$; to find x.

First $8x\div 2=4x$, and $4^2=16$; then $84+16=100=(x+4)^2$; then $\sqrt{100}=10=x+4$; then $10-4=6=x$.

$6^2=36+8\times 6=84.$ } Proof.

3. Given $3x^2+12x=288$; to find x.

First by dividing by 3, we have $x^2+4x=96$; then $4x\div 2=2x$, and $2^2=4$; then $96+4=100$; and $\sqrt{100}=10=x+2$; then $10-2=8=x$.

$8^2\times 3=192$, and $8\times 12=96$, and $192+96=288$. } Proof.

4. Given $4x^2+9x=405$ to; find x.

First by dividing by 4, we have $x^2+2\frac{1}{4}x=101\frac{1}{4}$; then $2\frac{1}{4}\div 2=1\frac{1}{8}$ or $=\frac{9}{8}$; and $\frac{9^2}{8}=\frac{81}{64}$; then $101\frac{1}{4}+\frac{81}{64}=102\frac{33}{64}$; then $\sqrt{102\frac{33}{64}}=\frac{81}{8}$ or $=10\frac{1}{8}$; then $10\frac{1}{8}-1\frac{1}{8}=9=x$.

$9^2=81\times 4=324$, and $9\times 9=81$; then $324+81=405$. } Proof.

5. Given $x^2-4x=60$; to find x.

First $4x\div 2=2x$; then $2^2=4$, and $60+4=64$, and $\sqrt{64}=8=x-2$; then $8+2=10=x$.

$10^2=100$, and $4\times 10=40$, and $100-40=60$. } Proof.

6. Given $10x^2-15x=100$; to find x.

First by dividing by 10, we have $x-1\frac{1}{2}x=10$; then $1\frac{1}{2}x\div 2=\frac{3}{4}x$; and $\frac{3^2}{4}=\frac{9}{16}$; then $10+\frac{9}{16}=\frac{169}{16}$, and $\sqrt{\frac{169}{16}}=\frac{13}{4}=3\frac{1}{4}$, and $3\frac{1}{4}+\frac{3}{4}=4=x$.

$4^2=16\times10=160$, and $4\times15=60$, and $160-60=100$. } Proof.

7. Given $2x^2-x=15$; to find x.

First by dividing by 2, we have $x^2-\frac{x}{2}=7\frac{1}{2}$; then one-half of $\frac{x}{2}=\frac{x}{4}$, or $\frac{1}{4}x$, and $\frac{1^2}{4}=\frac{1}{16}$; then $7\frac{1}{2}+\frac{1}{16}=\frac{121}{16}$, and $\sqrt{\frac{121}{16}}=\frac{11}{4}=2\frac{3}{4}$, and $2\frac{3}{4}+\frac{1}{4}=3=x$.

$3^2=9\times2=18-3=15$. } Proof.

EXAMPLES FOR PRACTICE.

8. Given $4x^2-x=248$; to find x.

Answer $x=8$.

9. Given $x^2+6x+20=335$; to find x.

Answer $x=15$.

10. Given $6x^2-10x+30=186$; to find x.

Answer $x=6$.

11. Given $x^2+8x=2x^2+12$; to find x.

First $2x^2-x^2=x^2$; then by transposition we have $x^2=8x-12$; therefore $x^2-8x=-12$; then by completing the square, we have $12-4^2=-4$; then $\sqrt{4}=2$, and $4+2=6=x$.

$6^2+6\times8=84$, and $2x^2+12=84$. } Proof.

12. Given $6x^2+4x=4x^2+198$; to find x.

First $6x^2-4x^2=2x^2$; we then have $2x^2+4x=198$; then by dividing by 2, and completing the square, &c., we have $x=9$.

13. Given $\frac{x^2}{2}+6x=80$; to find x.

First by multiplying by 2, we have $x^2+12x=160$; then by completing the square, &c. as before, we have $x=8$.

14. Given $\frac{x^2}{3}+x=60$; to find x.

First by multiplying by 3, we have $x^2+3x=180$.

Answer $x=12$.

15. Given $\frac{9x^2}{2}+6x=58\frac{1}{2}$; to find x.

First by dividing by $4\frac{1}{2}$, or $\frac{9}{2}$, we have $x^2+\frac{4x}{3}=13$; and then proceeding as before, we find $x=3$.

16. Given $\frac{3x^2}{2}+\frac{5x^2}{16}+10x-9=60$; to find x.

Answer $x=4$.

17. Given $ax^2+bx=c$; to find x.

First by dividing by a, we have $x^2+\frac{b}{a}x=\frac{c}{a}$; then $x=\sqrt{\frac{c}{a}+\left(\frac{b}{a}x\div 2\right)^2}-\frac{b}{a}x\div 2$; and by putting $a=6$, $b=12$, and $c=144$, we find $6x^2+12x=144$; then by proceeding as before, we have $x=4$.

18. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8$; to find x.

First $8-7\frac{3}{8}=\frac{5}{8}$; then by multiplying by 2, we have $x^2-\frac{2}{3}x=1\frac{1}{4}$; and proceeding as before, we find $x=1\frac{1}{2}$.

19. Given $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$; to find x.

First by multiplying by 2, we have $x-\frac{2}{3}\sqrt{x}=44\frac{1}{3}$; then $\frac{2}{3}\div 2=\frac{1}{3}$, and $\frac{1^2}{3}=\frac{1}{9}$; then $44\frac{1}{3}+\frac{1}{9}=44\frac{4}{9}$, by completing the square; then $\sqrt{44\frac{4}{9}}=6\frac{2}{3}$, and $6\frac{2}{3}+\frac{1}{3}=7=\sqrt{x}$, and $7^2=49=x$.

20. Given $5x-10\sqrt{x}=40$; to find x.

First by dividing by 5, we have $x-2\sqrt{x}=8$; then $2\sqrt{x}\div 2=\sqrt{x}$, and $1^2=1$; then $8+1=9$, by completing the square; then $\sqrt{9}=3=\sqrt{x}-1$, and $3+1=\sqrt{x}=4$; or $4^2=16=x$.

$$\left.\begin{array}{r}16\times 5=80;\ 10\times 4=40;\\ \text{and } 80-40=40.\end{array}\right\}\text{Proof.}$$

21. Given $6x-12\sqrt{x}=144$; to find x.

First by dividing by 6, we have $x-2\sqrt{x}=24$; then by completing the square, $24+1=25=(x-1)^2$; then $\sqrt{25}=5=x-1$, and $5+1=6=\sqrt{x}$, and $x=36$.

$$\left.\begin{array}{r}36\times 6=216-12\sqrt{x}=72,\\ \text{and } 216-72=144.\end{array}\right\}\text{Proof.}$$

22. Given $\left(\frac{12\sqrt{x}+x}{7}\right)=x$; to find x.

First $7x=12\sqrt{x}+x$; i.e. $6x=12\sqrt{x}$; then $12\div 6=2=\sqrt{x}$, or $x=4$.

23. Given $\left(\frac{10\sqrt{x}+2x}{4\frac{1}{2}}\right)=x$; i.e. $4\frac{1}{2}x=10\sqrt{x}+2x$; to find x.

Then $4\frac{1}{2}x-2x=2\frac{1}{2}x=10\sqrt{x}$; $10\div 2\frac{1}{2}=4=\sqrt{x}$, or $x=16$.

24. Given $x^2-10x=-24$; to find x.

First $10\div 2=5$, and $5^2=25$, and $25-24=1$, and $1+5=6=x$.

This case is the only one where the rules are not entirely different from any that I have seen given by others. I have endeavored to give this as simply as possible. I have also given a greater variety of examples in this case than any other, as it applies to a number of equations, many of which are solved with more ease by other rules, which will be given in their proper places.

CASE II.

The sum of two numbers, and their product given, to find those numbers.

NOTE.—In all problems of this kind, whether the product, the square, cube, or any higher power be involved, in two or any greater number of terms in arithmetical progression, the sum of the numbers, divided by the number of terms, will always quote a mean proportional to the other numbers involved; then, by finding a common difference, the unknown quantity or quantities will stand free from all other involvements. Therefore, in this case, observe this

RULE.

One-half of the given sum of the two numbers will be a mean proportional to the two required numbers; then the square of that proportional, less the given product, will equal the square of one-half the difference of the two required numbers. Therefore, let s=the sum of the two numbers, p=their product, x=the less and y=the greater, n=the number of terms, m=the mean proportional, q=the difference between the product and the square of the mean proportional, and $2d$=the difference of the two numbers. Then $s\div n=m$, and $m^2-p=q$, and $\sqrt{q}=d$; therefore $m-d=x$, and $m+d=y$.

EXAMPLES.

1. Given the sum of two numbers$=20=s$, and their product$=96=p$; to find those two numbers.

First $s\div n$ or $20\div 2=m$ or$=10$; then m^2 or $10^2=100$, and m^2-p or $100-96=4=d^2$; then $\sqrt{4}=2=d$; therefore $10-2=8=x$, and $10+2=12=y$, and the two required numbers are 8 and 12.

$8+12=20$, and $8\times 12=96$. $\}$ Proof.

2. Given the sum of two numbers, $s=26$, and their product $p=153$; to find those numbers.

First $s \div n = 13 = m$; then $m^2 = 169$, and $169 - 153 = 16 = q$; then $\sqrt{q} = 4 = d$; therefore $13 - 4 = 9 = x$, and $13 + 4 = 17 = y$, and the two required numbers are 9 and 17.

$9 + 17 = 26$, and $9 \times 17 = 153$. } Proof.

3. Given the sum of two numbers, $s = 91$, and their product $p = 2050$; to find those numbers.

First $s \div n = 45\frac{1}{2} = m$; then $m^2 = 2070\frac{1}{4}$; then $2070\frac{1}{4} - 2050 = 20\frac{1}{4} = d^2$; then $\sqrt{20\frac{1}{4}} = 4\frac{1}{2} = d$; therefore $m - d = 41 = x$, and $m + d = 50 = y$; and the two required numbers are 41 and 50.

$41 + 50 = 91$, and $41 \times 50 = 2050$. } Proof.

4. Given the sum of two numbers, $s = 73$, and their product $p = 1332$; to find those numbers.

First $s \div n = 36\frac{1}{2} = m$; then $m^2 = 1332\frac{1}{4}$, and $1332\frac{1}{4} - 1332 = \frac{1}{4} = d^2$, and $\sqrt{\frac{1}{4}} = \frac{1}{2} = d$; therefore $36\frac{1}{2} - \frac{1}{2} = 36 = x$, and $36\frac{1}{2} + \frac{1}{2} = 37 = y$; and the two required numbers are 36 and 37.

Or problems of this kind may be solved thus:

5. Given the sum of two numbers $= 20$, and their product $= 64$; to find those numbers.

First $20 \div 2$ or $s \div n = 10 = m$; then $\overline{10 - d} \times \overline{10 + d} = 64$, by the question; then $\overline{10 - d} \times \overline{10 + d} = 100 - 10d + 10d - d^2$; then by cancelling $10d$, which is common to both sides of the equation, we have $100 - d^2 = 64$; then $100 - 64 = 36 = d^2$, and $\sqrt{36} = 6 = d$; therefore $10 - 6 = 4 = x$, and $10 + 6 = 16 = y$, and the two required numbers are 4 and 16.

$16 + 4 = 20$, and $16 \times 4 = 64$. } Proof.

6. It is required to divide the number 60 into two such parts that their product shall be $= 756$.

First $60 \div 2$ or $s \div n = 30 = m$; then $\overline{30 - d} \times \overline{30 + d} = 756$, by the question; and $\overline{30 - d} \times \overline{30 + d} = 900 - 30d + 30d - d^2$; then $30d$ being common to both sides of the equation, we have $900 - d^2 = 756$; then $900 - 756 = 144 = d^2$, and $\sqrt{144} = 12 = d$; therefore $30 - 12 = 18 = x$, and $30 + 12 = 42 = y$; and the two required numbers are 18 and 42.

$18 + 42 = 60$, and $18 \times 42 = 756$. } Proof.

Note.—Problems of the foregoing description may be solved without any data except the product; for any number, the square of which is greater than the product, and leaves a difference between that square and the given pro-

duct equal to a square number, is a mean proportional; but in this case they are not limited, and will frequently admit of different answers. Thus:

Given the product of two numbers $=240$; to find those numbers.

First we find $16^2=256-240=16$, which is a square number; then $256-240=16=d^2$; then $\sqrt{16}=4=d$; therefore $16-4=12=x$, and $16+4=20=y$; and 12 and 20 are two numbers whose product is 240.

Again: $19^2=361-240=121$, and $\sqrt{121}=11=d$; then put $19=m$, and $19-11=8=x$, and $19+11=30=y$; and 8 and 30 are two other numbers whose product is 240.

Again: $23^2=529-240=289$, and $\sqrt{289}=17=d$; then put $23=m$, and $23-17=6=x$, and $23+17=40=y$; and 6 and 40 are two other numbers whose product is 240.

Again: $41\frac{1}{2}^2=1722\frac{1}{4}-240=1482\frac{1}{4}$, and $\sqrt{1482\frac{1}{4}}=38\frac{1}{2}=d$; then put $41\frac{1}{2}=m$, and $41\frac{1}{2}-38\frac{1}{2}=3=x$, and $41\frac{1}{2}+38\frac{1}{2}=80=y$; and 3 and 80 are two other numbers whose product is 240. Thus also we shall find 61^2 gives $2\times120=240$, by pursuing the same process.

CASE III.

The difference of two numbers and their product given; to find those numbers.

RULE.

Put $x=$the less and $2d=$their difference; then will $x\times\overline{x+2d}=x^2+2dx$; then proceed as in case second.

EXAMPLES.

1. Given the difference of two numbers $=6$, and their product $=135$; to find those numbers.

First $x=$the less, and $x+6=$the greater; then $x\times\overline{x+6}=x^2+6x=135$, by the question; then $6\div2=3$, and $3^2=9$, and $135+9=144=m^2$; then $\sqrt{144}=12=m$; then $12-3=9=x$, and $12+3=15=y$; and the two required numbers are 9 and 15.

Or these problems may be solved thus:

The number whose square is more than 135, and will leave a difference equal to a square number, is $12^2=144$; then $144-135=9$, and $\sqrt{9}=3$; then put $12=m$, and $3=d$; and we have $12-3=9$, and $12+3=15$, as before.

2. Given the difference of two numbers$=9\frac{1}{2}$, and their product$=329$; to find those numbers.

First $x\times x+9\frac{1}{2}=x^2+9\frac{1}{2}x=329$, by the question; then $9\frac{1}{2}\div2=4\frac{3}{4}$, and $4\frac{3}{4}^2=22\frac{9}{16}$; then $329+22\frac{9}{16}=351\frac{9}{16}$; then $\sqrt{351\frac{9}{16}}=18\frac{3}{4}=m$; then $18\frac{3}{4}-4\frac{3}{4}=14=x$, and $14+9\frac{1}{2}=23\frac{1}{2}=y$, or $18\frac{3}{4}+4\frac{3}{4}=23\frac{1}{2}=y$; and the two required numbers are 14 and $23\frac{1}{2}$.

EXAMPLES.

1. Given the sum of two numbers$=21$, and their product $=106\frac{1}{4}$; to find those two numbers.

Answer $8\frac{1}{2}$ and $12\frac{1}{2}$.

2. Given the sum of two numbers$=40$, and the product of one-fourth of each of them$=24$; to find those numbers.

First $\frac{1^2}{4}=\frac{1}{16}$; then $24\times16=384$, and $384\div1$ or the square of the numerator$=384=$the product of two numbers whose sum is 40; we then have $s=40$, and $p=384$, to find x and y, as before. Answer 16 and 24.

$\frac{1}{4}$ of $16=4$, and $\frac{1}{4}$ of $24=6$, and $4\times6=24$. } Proof.

3. Given the sum of two numbers$=24$, and the product of one-third of each of them$=15$; to find those numbers.

First $\frac{1^2}{3}=\frac{1}{9}$; then $15\times9=135=$the product of two numbers whose sum is 24; we then have given the sum of two numbers$=24$, and their product$=135$. Answer 9 and 15.

$\frac{1}{3}$ of $9=3$, $\frac{1}{3}$ of $15=5$, and $3\times5=15$. } Proof.

4. Given the sum of two numbers$=48$, and the product of three-fourths of each of them$=243$; to find those numbers.

First $\frac{3^2}{4}=\frac{9}{16}$; then $243\times16=3888$, and $3888\div9=432$ $=$the product of two numbers whose sum is 48; we then have given the sum of two numbers$=48$, and their product $=432$, to find x and y. Answer 12 and 36.

5. Given the sum of two numbers$=35$, and the product of four-sevenths of each of them$=96$; to find those numbers.

First $\frac{4^2}{7}=\frac{16}{49}$; then $96\times49=4704$, and $4704\div16=294=$ the product of two numbers whose sum is 35; we then have given $s=35$, and $p=294$, to find x and y.

Answer 14 and 21.

6. Given the sum of two numbers $x+y=48$, and the

product of three-fifths of x and four-sevenths of $y=192$; to find those two numbers.

First $\frac{3}{5}\times\frac{4}{7}=\frac{12}{35}$; then $192\times35=6720$, and $6720\div12=560=xy$; we then have given $s=48$, and $p=560$, to find x and y. Answer 20 and 28.

7. It is required to divide the number 60 into two such parts that the product of two-thirds of each of them shall be $=384$.

First $\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$; then $384\times9=3456$, and $3456\div4=864$; we then have $s=60$, and $p=864$, to find x and y.
Answer 24 and 36.

8. It is required to divide the number 80 into two such parts that their product shall be $=1536$.
Answer 32 and 48.

CASE IV.

The sum of three numbers in arithmetical progression and their continued product given; to find those numbers.

RULE.

Find a mean proportional, or the middle term, by dividing the given sum by the number of terms; then the cube of that mean proportional, less the product, will be equal to the square of the difference, multiplied by the mean proportional. Or thus:

Let x equal the least, m the middle, and y the greatest of the three numbers; s equal their sum, p their product, q the difference between their product and m^3; n equal the number of terms, and d the common difference; then will $s\div n=m$, and $m^3-p=md^2$ or q; then $q\div m=d^2$; then $m-d=x$, and $m+d=y$.

EXAMPLES.

1. Given the sum of three numbers in arithmetical progression equal 12, and their continued product equal 48; to find those numbers.

First $12\div3$ or $s\div n=4=m$; then $m^3=64$, and $m^3-p=16=q=md^2$; then $q\div m=4=d^2$, and $\sqrt{4}=2=d$; therefore $4-2=2=x$, $4=m$, and $4+2=6=y$; and the three required numbers are 2, 4 and 6.

$2+4+6=12$, and $2\times4\times6=48$. } Proof.

Or thus: Having found $m=4$, we then have $(4-d)\times4$

$\times(4+d)$; i. e. $=64-16d+16d-4d^2=48$; then by cancelling $16d$, we have $64-4d^2=48$; then $64-48=16=4d^2$, and $16\div4=4=d^2$, and $\sqrt{4}=2=d$; and the answer is, as before, 2, 4 and 6.

2. Given the sum of three numbers in arithmetical progression, $s=48$, and their continued product $p=3312$; to find those numbers.

First $s\div n=16=m$, and $m^3=4096$; then $4096-p=784=md^2$; then $784\div m=49=d^2$, and $\sqrt{49}=7=d$; therefore $16-7=9=x$, $16=m$, and $16+7=23$; and the three required numbers are 9, 16 and 23.

$9+16+23=48$, and $9\times16\times23=3312$. } Proof.

Or thus: Having found $m=16$, then $(16-d)\times16\times(16+d)=m^3$ or $=4096-16d^2=3312$; therefore $4096-3312=784=16^2d$; then $784\div16=49=d^2$; and the three required numbers are, as before, 9, 16 and 23.

3. Given the sum of three numbers in arithmetical progression equal 39, and the product of the two first, or xm, equal 117; to find those three numbers.

First $s\div n=13=m$; then $117\div13=9=x$; and having found m to be $=13$, the common difference or $d=4$, therefore the three required numbers of course are 9, 13 and 17.

Or thus: Having found $m=13$, we then have by the question $13-d\times13$, or $m-d\times m=117$; then $m^2=169=md+117$; then $169-117=52=md$; then $52\div m=4=d$; and the three required numbers are, as before, 9, 13 and 17.

4. Given the sum of three numbers in arithmetical progression equal 54, and the product of the two last, or my, equal 432; to find those three numbers.

First $54\div3=18=m$; then $432\div m=24=y$; and having found m or the middle number $=18$, gives the common difference, or $y-m=6$; and the three required numbers are 12, 18 and 24.

Or thus: Having found $m=18$, we have by the question $m\times m+d=432$; i. e. $18^2=324+18d=432$; then $432-324=108=18d$; then $108\div18=6=d$; and the three required numbers, as before, are 12, 18 and 24.

5. Given the sum of three numbers in arithmetical progression equal 45, and the product of the two extremes equal 200; to find those three numbers.

First 45 or $s\div n=15=m$; then $m^2=225-p=d^2$; then $225-200=25=d^2$; then $\sqrt{25}=5=d$; thus having found m and the common difference, the three required numbers are 10, 15 and 20.

Or thus: Having found $m=15$, we have by the question $\overline{15-d}\times\overline{15+d}=200$; i. e. $15^2=225-d^2=200$; then $225-200=25=d^2$; and $\sqrt{25}=5=d$, as before; and the three required numbers are 10, 15 and 20.

6. Given the sum of three numbers in arithmetical progression equal 54, and the continued product of one-third of each equal 192; to find those three numbers.

First $\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{1}{27}$; then $192\times27=5184$; we then have given the sum, or $s=54$, and the continued product of the three numbers $=p=5184$; and by proceeding as before, we find the three required numbers to be 12, 18 and 24.

7. Given the sum of three numbers in arithmetical progression equal 60, and the continued product of one-third of the first, three-fourths of the second, and four-fifths of the third equal 1500; to find those three numbers.

First $\frac{1}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{12}{60}=\frac{1}{5}$; then $1500\times5=7500=$ the continued product of three numbers whose sum is 60; we then have given the sum of three numbers in arithmetical progression $=60$, and their continued product $=7500$, to find those numbers. Answer 15, 20 and 25.

8. Given the sum of three numbers in arithmetical progression equal 84, and the continued product of three-fifths of the first, four-sevenths of the second, and five-ninths of the third equal 3840; to find those three numbers.

First $\frac{3}{5}\times\frac{4}{7}\times\frac{5}{9}=\frac{60}{315}=\frac{12}{63}$ or $=\frac{4}{21}$; then $3840\times21=80640$, and $80640\div4=20160=$ the continued product of three numbers whose sum is 84. Answer 20, 28 and 36.

CASE V.

The sum of four numbers in arithmetical progression and their continued product being given; to find those numbers.

RULE.

Find a mean proportional, as before, by dividing the sum by the number of terms; then the product of the two means, less twice the square of the common difference, is equal to the product of the two extremes; and as the product of the two extremes, multiplied by the product of the two means, is equal to the continued product of the four numbers, we

have the product of two numbers whose difference is equal to twice the square of the common difference of the four required numbers, equal to the continued product of the four numbers; then by case second find a number, the square of which is greater than the given product, and leaves a difference between that square and the given product equal to a square number; then the square root of that number will be equal to half their difference. We then have the product of the means and the two extremes; and as the mean proportional to the two means will also be a mean proportional to the two extremes, and likewise to the four required numbers, they may be easily found by case second.

Or thus: Let the four numbers be equal $wxyz$; let s equal their sum, p their product, pe the product of the two extremes, and pm the product of the two means; n equal the number of terms, m the mean proportional, q the difference between m^2 and pe and pm, and $2d$ the common difference of the four numbers; then will $s \div n = m$, and $m^2 - pe$ or $pm = d^2$.

EXAMPLES.

1. Given the sum of four numbers in arithmetical progression equal 20, and their continued product equal 384; to find those numbers.

First a few trials will show 20^2 to be greater than 384, and leave a difference equal to a square number; then $20^2 = 400 = p + d^2$; then $400 - 384 = 16 = d^2$, and $\sqrt{16} = 4 = d$; then $20 - d = 16 = pe$, and $20 + 4 = 24 = pm$; we then have the product of the two means and the two extremes, and $s \div n = 5 = m$, or a mean proportional to each; then $5^2 = 25$, and $25 - pm = 1 = d^2$, and $\sqrt{1} = 1 = d$; then $5 - 1 = 4 = x$, and $5 + 1 = 6 = y$; and thus having found the two means, we have the four required numbers. But the two extremes may also be found thus: $5^2 = 25 = m^2$; then $25 - 16 = 9 = d^2$, and $\sqrt{9} = 3$; then $5 - 3 = 2$, and $5 + 3 = 8 =$ the two extremes, or $= w$ and z; we then have the four required numbers $= 2$, 4, 6 and 8.

2. Given the sum of four numbers in arithmetical progression equal 26, and their continued product equal 880; to find those numbers.

First a few trials will show that 31^2 is greater than 880, and leaves $q =$ to a square number; then $31^2 = 961 - 880 = 81 = q$; then $\sqrt{q} = 9$, and $31 - 9 = 22 = pe$, and $31 + 9 = 40 = pm$; then $s \div n$ or $26 \div 4 = 6\frac{1}{2} = m$, and $m^2 = 42\frac{1}{4}$, and $42\frac{1}{4} - 22 = 20\frac{1}{4} = d^2$, and $\sqrt{20\frac{1}{4}} = 4\frac{1}{2} = d$; then $6\frac{1}{2} - 4\frac{1}{2} = 2$

$=w$, and $6\frac{1}{2}+4\frac{1}{2}=11=z$; we then have the two extremes $=2$ and 11, and a mean proportional$=6\frac{1}{2}$; which determines the four numbers to be 2, 5, 8 and 11.

Or the two means may be found thus: $m^2=42\frac{1}{4}-pm$, or $40=2\frac{1}{4}=d^2$, and $\sqrt{2\frac{1}{4}}=1\frac{1}{2}=d$; then $6\frac{1}{2}-1\frac{1}{2}=5=x$, and $6\frac{1}{2}+1\frac{1}{2}=8=y$: and thus also we find the four required numbers to be 2, 5, 8 and 11.

3. Given the sum of four numbers in arithmetical progression equal 36, and their continued product equal 3495; to find those numbers.

First by a few trials we shall find $61^2=3721-p=256=q$, which is a square number; then $\sqrt{q}=16=d$, and $61-16=45=pe$, and $61+16=77=pm$; we then have the product of the two means$=77$, and the product of the two extremes$=45$; then $s\div n=9=m$, and $9^2=81=pm+d^2$; then $81-pm=4=d^2$, and $\sqrt{4}=2=d$; then $9-2=7=x$, and $9+2=11=y$. Thus having found the two means and the common difference, the four required numbers are 3, 7, 11 and 15. Or $81-pe=36$, and $\sqrt{36}=6=d$; then $9-6=3=w$, and $9+6=15=z$, as before.

4. Given the product of the two means of four numbers in arithmetical progression equal 35, and the continued product of the said four numbers equal 945; to find those numbers.

First $945\div35=27=$the product of the two extremes; we then have the product of the two extremes$=27$, and of the two means$=35$; and by pursuing the former process, we find the four required numbers to be 3, 5, 7 and 9.

5. Given the difference of the two extremes of four numbers in arithmetical progression equal 9, and the product of the said two numbers equal 70; to find the four numbers.

Let $w=$the less; then will $w+9=$the greater; then will $w\times w+9$ or $w^2+9w=70$, by the question; then by case first we shall find $w=5$, and $w+9=14$; and thus having found the two extremes and the difference$=9$, then $9\div3=3=$the common difference of the four required numbers; and we find the four numbers to be 5, 8, 11 and 14.

CASE VI.

The sum of two numbers and the sum of their squares given; to find those numbers.

RULE.

One-half the sum will be a mean proportional, and one-

half the sum of the squares is equal to the square of the mean proportional, more the square of one-half the difference of the two required numbers; or the sum of the two squares, less twice the square of the mean proportional, is equal to twice the square of one-half the difference of the two numbers. Or thus: Let s equal the sum, s^2 the sum of the squares, n the number of terms, m the mean proportional, x the less, y the greater, and $2d$ the difference of the two numbers; then will $s \div n = m$, $s^2 \div n = m^2 + d^2$, or $(m-d)^2 + (m+d)^2 = s^2 + 2d^2$.

EXAMPLES.

1. Given the sum of two numbers equal 16, and the sum of their squares equal 160; to find those numbers.

First $s \div n = m = 8$, and $s^2 \div n = 80 = m^2 + d^2$; then $m^2 = 64$, and $80 - 64 = 16 = d^2$, and $\sqrt{16} = 4 = d$; therefore $8 - 4 = 4 = x$, and $8 + 4 = 12 = y$; and the two required numbers are 4 and 12.

Or thus: Having found $m = 8$, then $(8-d)^2 + (8+d)^2 = 160$; i. e. $2m^2 + 2d^2 = 160$; then $2m^2 = 128$, and $160 - 126 = 32 = 2d^2$, and $32 \div 2 = 16 = d^2$, and $\sqrt{16} = 4 = d$; therefore $8 - 4 = 4 = x$, and $8 + 4 = 12 = y$; and the two required numbers are, as before, 4 and 12.

2. Given the sum of two numbers equal 19, and the sum of their squares equal 185; to find those numbers.

First $19 \div 2 = 9\frac{1}{2} = m$, and $185 \div 2 = 92\frac{1}{2} = m^2 + d^2$; then $m^2 = 90\frac{1}{4}$, and $92\frac{1}{2} - 90\frac{1}{4} = 2\frac{1}{4} = d^2$, and $\sqrt{2\frac{1}{4}} = 1\frac{1}{2} = d$; then $9\frac{1}{2} - 1\frac{1}{2} = 8$, and $9\frac{1}{2} + 1\frac{1}{2} = 11$; and the two required numbers are 8 and 11.

Or thus: $(9\frac{1}{2} - d)^2 + (9\frac{1}{2} + d)^2 = 185$; i. e. $2m^2 + 2d^2 = 185$; then $2m^2 = 180\frac{1}{2}$, and $185 - 180\frac{1}{2} = 4\frac{1}{2} = 2d^2$, and $4\frac{1}{2} \div 2 = 2\frac{1}{4} = d^2$, and $\sqrt{2\frac{1}{4}} = 1\frac{1}{2}$; and the two required numbers are, as before, 8 and 11.

$8 + 11 = 19$, and $8^2 + 11^2 = 185$. } Proof.

3. Given the sum of two numbers equal $10\frac{1}{2}$, and the sum of their squares equal $63\frac{1}{8}$; to find those numbers.

First $s \div n = 5\frac{1}{4} = m$, and $s^2 \div n = 31\frac{9}{16}$; then $m^2 = 27\frac{9}{16}$, and $31\frac{9}{16} - 27\frac{9}{16} = 4 = d^2$, and $\sqrt{4} = 2 = d$; therefore $5\frac{1}{4} - 2 = 3\frac{1}{4} = x$, and $5\frac{1}{4} + 2 = 7\frac{1}{4} = y$; and the two required numbers are $3\frac{1}{4}$ and $7\frac{1}{4}$.

Or thus: $(5\frac{1}{4} - d)^2 + (5\frac{1}{4} + d)^2 = 63\frac{1}{8}$, by the question; i. e. $2m^2 + 2d^2 = 63\frac{1}{8}$; then $2m^2 = 55\frac{1}{8}$, and $63\frac{1}{8} - 55\frac{1}{8} = 8 = 2d^2$; then $8 \div 2 = 4 = d^2$, and $\sqrt{4} = 2 = d$; and the two required numbers are, as before, $3\frac{1}{4}$ and $7\frac{1}{4}$.

The difference of two numbers and the sum of their squares given; to find those numbers.

Observe the same symbols as before.

4. Given the difference of two numbers equal 4, and the sum of their squares equal 136; to find those numbers.

First $s^2\times2=272$, and $4^2=16$; then $272-16=256=(x+y)^2$; then $\sqrt{256}=16=x+y$, or equal the sum of the two numbers; we then have given the sum of two numbers equal 16, and the sum of their squares equal 136; then by proceeding as before, we find $x=6$, and $y=10$; by which we find this theorem: $2s^2-d^2$, or the square of the given difference $=\square\, s$, or equal the square of the sum of the two required numbers.

Or thus: Let $x=$ the less, then will $x+4=$ the greater; then by the question, $x^2+(x+4)^2=136$; then let $2d=$ the given difference, and $2(x+d)^2+2d^2=136$; then $136\div2=68=(x+d)^2+d^2$; then as d is equal one-half of 4, or $d=2$, and $2^2=4$, then $68-4=64=(x+d)^2$, and $\sqrt{64}=8=x+2$, or $8-2=x$; then $6=x$, and $6+4=10=y$; and the two required numbers, as before, are 6 and 10.

$10-6=4$, and $6^2+10^2=136$. } Proof.

5. Given the difference of two numbers equal 6, and the sum of their squares equal 218; to find those numbers.

First s^2 or $218\times2=436$, and $436-6^2=400=(x+y)^2$; then $\sqrt{400}=20=x+y$, or equal the sum of the two numbers; we then have given $s=20$, and $s^2=218$, to find the two numbers; then by proceeding as before, we find $x=7$, and $y=13$.

Or thus: Let $x=$ the less, then will $x+6=$ the greater, and $2d=$ the difference; then will $2(x+d)^2+2d^2=218$; then $218\div2=109$; and as d is given $=3$, and $3^2=9$, then $109-9=100=(x+d)^2$; then $\sqrt{100}=10=x+d$; then $10-3=7=x$, and $7+6=13=x+6$; and the two required numbers, as before, are 7 and 13.

6. Given the sum of two numbers equal 22, and the difference of their squares equal 132; to find those numbers.

First $132\div22=6$, the difference of the two required numbers; we then have the sum of two numbers equal 22, and their difference equal 6, to find those two numbers; then $22\div2$ or $s\div n=11=m$, and $6\div2=3=d$; then $m-d=8=x$, and $m+d=14=y$; and the two required numbers are 8 and 14. $8+14=22$, and $14^2-8^2=132$. } Proof.

Or may be stated thus:

Given $x+y=22$, and $y^2-x^2=132$, and $132\div22=y-x=6$.

7. Given the sum of two numbers equal 17, and the difference of their squares equal 136; to find those numbers.

First $136 \div 17 = 8$, the difference of the two required numbers; we then have $s=17$, and $y-x=8$; then $s \div n = 8\frac{1}{2}$, and $8 \div 2 = 4 = d$; therefore $8\frac{1}{2} - 4 = 4\frac{1}{2} = x$, and $8\frac{1}{2} + 4 = 12\frac{1}{2} = y$; and the two required numbers are $4\frac{1}{2}$ and $12\frac{1}{2}$.

8. Given the difference of two numbers equal 3, and the difference of their squares equal 63; to find those numbers.

First $63 \div 3 = 21$, the sum of the two required numbers; we then have the sum of two numbers equal 21, and their difference equal 3; then $21 \div 2 = 10\frac{1}{2} = m$, and $3 \div 2 = 1\frac{1}{2} = d$; then $10\frac{1}{2} - d = 9 = x$, and $10\frac{1}{2} + d = 12 = y$; and the two required numbers are 9 and 12.

9. Given the difference of two numbers equal 6, and the difference of their squares equal 84; to find those numbers.

First $84 \div 6 = 14$, the sum of the two required numbers; we then have the sum of two numbers equal 14, and their difference equal 6; then $14 \div 2 = 7 = m$, and $6 \div 2 = 3 = d$; therefore $7 - d = 4 = x$, and $7 + d = 10 = y$; and the two required numbers are 4 and 10.

$10 - 4 = 6$, and $10^2 - 4^2 = 84$. } Proof.

10. Given the sum of two numbers, more the difference of their squares equal 40; to find those two numbers.

First find a number the square of which is greater than 40, and leaves a difference equal to a square number; this by a few trials will be found to be $6\frac{1}{2}$; then $6\frac{1}{2}^2 = 42\frac{1}{4}$, and $42\frac{1}{4} - 40 = 2\frac{1}{4}$, and $\sqrt{2\frac{1}{4}} = 1\frac{1}{2}$; then $6\frac{1}{2} + 1\frac{1}{2} = 8$, the sum of the two numbers; then $40 - 8 = 32$, the difference of their squares; we then have given the sum of two numbers equal 8, and the difference of their squares equal 32; then $32 \div 8 = 4$, the difference of the two numbers; then $8 \div 2 = 4 = m$, and $4 \div 2 = 2 = d$; therefore $4 - 2 = 2 = x$, and $4 + 2 = 6 = y$; and the two required numbers are 2 and 6.

$6 + 2 = 8$, and $6^2 - 2^2 = 32$, and $32 + 8 = 40$. } Proof.

11. Given the sum of two numbers, more the difference of their squares equal 98; to find those numbers.

A few trials will show that $10\frac{1}{2}^2 = 110\frac{1}{4}$, will leave a difference between 98 equal a square number; then $110\frac{1}{4} - 98 = 12\frac{1}{4}$, and $\sqrt{12\frac{1}{4}} = 3\frac{1}{2}$; then $10\frac{1}{2} + 3\frac{1}{2} = 14 = s$, or the sum of the two required numbers; then $98 - 14 = 84$, the difference of their squares; we then have the sum of two numbers equal 14, and the difference of their squares equal 84; then $84 \div 14 = 6$, the difference of the two required numbers; then $14 \div 2 = 7 = m$, and $6 \div 2 = 3 = d$; then $m - d =$

$4=x$, and $m+d=10=y$; and the two required numbers are 4 and 10.

12. Given the sum of two numbers, more the sum of their squares equal 68; to find those numbers.

First $68\div2=34=m^2+m+d^2$; then the number, the square of which is less than 34, and leaves a difference equal to a square number, is $5^2=25$; then put $5=m$, and $34-5=29=m^2+d$; then $29-25=4=d^2$, and $\sqrt{4}=2=d$; then $5-2=3=x$, and $5+2=7=y$; and the two required numbers are 3 and 7.

$7+3=10$, and $7^2+3^2=58$, and $58+10=68$. } Proof.

13. Given the sum of two numbers, more the sum of their squares equal 278; to find those two numbers.

First $278\div2=139=m^2+m+d^2$; then it may be easily found that 9^2, after deducting 9 from 139, will leave a square number; then put $9=m$, and $139-9=130=m^2+d^2$; then $9^2=81$, and $130-81=49=d^2$; then $\sqrt{49}=7=d$; therefore $9-7=2=x$, and $9+7=16=y$; and the two required numbers are 2 and 16.

$2+16=18$, and $2^2=4$, and $16^2=256$, and $256+4=260+18=278$. } Proof.

14. Given the sum of two numbers equal 20, and the sum of the squares of one-half of each of them equal 52; to find those numbers.

First $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$; then $52\times4=208=s^2$, or the sum of the squares of the two numbers; we then have $x+y=20=s$, and $x^2+y^2=208$; then by proceeding as before, we find $x=8$ and $y=12$; and the two required numbers are 8 and 12.

15. Given the sum of two numbers equal 40, and the sum of the squares of one-fifth of each of them equal 34; to find those two numbers.

First $\frac{1}{5}\times\frac{1}{5}=\frac{1}{25}$, and $34\times25=850=x^2+y^2$ or $=s^2$; we then have given $s=40$, and $s^2=850$; and by proceeding as before, we find $x=15$, and $y=25$.

CASE VII.

The sum of three numbers in arithmetical progression and the sum of their squares being given; to find those numbers.

RULE.

The sum of the three numbers, divided by 3, will quote a mean proportional, or the middle number; then the square of the middle number, deducted from one-third of the sum of the squares, leaves two-thirds of the square of the common difference. Or thus: Let x equal the least of the three numbers, m the middle number, and y the greatest; s equal the sum, s^2 the sum of the squares, n the number of terms, q the difference between m^2 and one-third s^2, and d the common difference; then will $s \div n = m$, and $s^2 \div n - m^2 = q$, and $3q \div 2 = d^2$, or $s^2 - 3m^2 = 2d^2$; then $m - d = x$, and $m + d = y$.

EXAMPLES.

1. Given the sum of three numbers in arithmetical progression equal 12, and the sum of their squares equal 56; to find those numbers.

First $s \div n = 4 = m$, and $s^2 \div n = 18\frac{2}{3} = m^2 + \frac{2}{3}d^2$; then $m^2 = 16$, and $18\frac{2}{3} - 16 = 2\frac{2}{3} = q$, and $3q \div 2 = 4 = d^2$, and $\sqrt{4} = 2 = d$; then $4 - d = 2$, and $4 + d = 6$; and the three required numbers are 2, 4 and 6.

Or thus: Having found m, we have $(4-d)^2 + 4^2 + (+d)^2 = 56$; i. e. $3m^2 + 2d^2 = 56$; then $3m^2 = 48$, and $56 - 48 = 8 = 2d^2$; then $8 \div 2 = 4 = d$, and $\sqrt{4} = 2 = d$; and we find the three numbers, as before, are 2, 4 and 6.

2. Given the sum of three numbers in arithmetical progression equal 48, and the sum of their squares equal 886; to find those three numbers.

First $48 \div 3 = 16 = m$, and $866 \div 3 = 288\frac{2}{3} = m^2 + \frac{2}{3}d^2$; then $m^2 = 256$, and $288\frac{2}{3} - 256 = 32\frac{2}{3} = q$, and $3q \div 2 = 49 = d^2$, and $\sqrt{49} = 7 = d$; therefore $16 - 7 = 9 = x$, and $16 + 7 = 23 = y$; and the three required numbers are 9, 16 and 23.

Or thus: Having found $m = 16$, we have by the question $(16-d)^2 + 16^2 + (16+d)^2 = 886$; i. e. $3m^2 + 2d^2 = 886$; then $3m^2 = 768$, and $866 - 768 = 98 = 2d^2$; then $98 \div 2 = 49 = d^2$, and $\sqrt{49} = 7 = d$; and we find the three required numbers, as before, are 9, 16 and 23.

3. Given the sum of three numbers in arithmetical progression equal $19\frac{1}{2}$, and the sum of their squares equal $134\frac{3}{4}$; to find those three numbers.

First $19\frac{1}{2} \div 3 = 6\frac{1}{2} = m$, and $134\frac{3}{4} \div 3 = 44\frac{11}{12} = m^2 + \frac{2}{3}d^2$; then $m^2 = 42\frac{1}{4}$, and $44\frac{11}{12} - 42\frac{1}{4} = 2\frac{2}{3} = q$, and $3q \div 2 = 4 = d^2$, and $\sqrt{4} = 2 = d$; therefore $6\frac{1}{2} - d = 4\frac{1}{2} = x$, and $6\frac{1}{2} + d = 8\frac{1}{2} = y$; and the three required numbers are $4\frac{1}{2}$, $6\frac{1}{2}$ and $8\frac{1}{2}$.

4. Given the sum of the squares of the two extremes of three numbers in arithmetical progression equal 116; to find those three numbers.

First $116\div2=58=m^2+d^2$; then the number, the square of which is less than 58, and will leave a difference q equal to a square number, is 7; then put $7=m$, and $7^2=49=m^2$; then $58-49=9=d^2$, and $\sqrt{9}=3$; therefore $7=m$, $7-3=4=x$, and $7+3=10=y$; and the three required numbers are 3, 7 and 10.

5. Given the sum of three numbers in arithmetical progression equal 18; and the sum of the squares of the two least of them equal 45; to find those three numbers.

First $45\div2=22\frac{1}{2}$; then the number, the square of which is less than $22\frac{1}{2}$, and will leave a difference q equal to a square number, is $4\frac{1}{2}$; then $4\frac{1}{2}^2=20\frac{1}{4}$, and $22\frac{1}{2}-20\frac{1}{4}=2\frac{1}{4}=d^2$, and $\sqrt{2\frac{1}{4}}=1\frac{1}{2}=d$; therefore $4\frac{1}{2}-1\frac{1}{2}=3=x$, and $4\frac{1}{2}+1\frac{1}{2}=6=m=\frac{1}{3}$ of s; therefore the three required numbers are 3, 6 and 9.

6. Given the sum of three numbers in arithmetical progression equal s, more the sum of their squares equal $s^2=68$; to find those three numbers.

First $68\div3=22\frac{2}{3}=m^2+m+\frac{2}{3}d^2$; then by a few trials we shall find $m=4$; then assuming $4=m$, $22\frac{2}{3}-4=18\frac{2}{3}=m^2+\frac{2}{3}d^2$; then $m^2=16$, and $18\frac{2}{3}-16=2\frac{2}{3}=\frac{2}{3}d^2$ or q; then $3q\div2=4=d^2$, and $\sqrt{4}=2=d$; therefore $4-2=2=x$, and $4+2=6=y$; and the three required numbers are 2, 4 and 6.

7. Again: Let the sum of three numbers in arithmetical progression, more the sum of their squares, be given, equal 186; to find those three numbers.

First $186\div3=62=m^2+m+\frac{2}{3}d^2$; then the number, the square of which is less than 62, and, deducted from 62, will leave two-thirds of a square number, will easily be found to be 7; then put $7=m$, and $62-7=55=m^2+\frac{2}{3}d^2$; then $55-m^2=6=\frac{2}{3}d^2$, and $6+3=9=d^2$; then $\sqrt{9}=3=d$; therefore $7-3=4=x$, and $7+3=10=y$; and the three required numbers are 4, 7 and 10.

CASE VIII.

The sum of four numbers in arithmetical progression and the sum of their squares being given; to find those numbers.

RULE.

Let w, x, y and z equal the four numbers, m a mean proportional, n the number of terms, s the sum, s^2 the sum of

the squares, q the difference between m^2 and $s^2 \div n$, and $2d$ the common difference of the four numbers; then will $s \div n = m$, and $s^2 \div n = m^2 + 5d^2$; or $s \div n - m^2 = q$, and $q \div 5 = d^2$; then $m - d = x$, and $m + d = y$; or $s^2 = 4m^2 + 5d^2$.

EXAMPLES.

1. Given the sum of four numbers in arithmetical progression $s = 14$, and the sum of their squares $s^2 = 54$; to find those four numbers.

First $s \div n = 3\frac{1}{2} = m$, and $s^2 \div n = 13\frac{1}{2} = m^2 + 5d^2$; then $m^2 = 12\frac{1}{4}$, and $13\frac{1}{2} - 12\frac{1}{4} = 1\frac{1}{4} = 5d^2$; then $1\frac{1}{4} \div 5 = \frac{1}{4} = d^2$, and $\sqrt{\frac{1}{4}} = \frac{1}{2} = d$; therefore $3\frac{1}{2} - \frac{1}{2} = 3 = x$, and $3\frac{1}{2} + \frac{1}{2} = 4 = y$; thus having found the two means to be 3 and 4, and the common difference 1, we have the four required numbers 2, 3, 4 and 5.

Or thus: $m^2 = 12\frac{1}{4}$; then $4m^2 = 49$, and $54 - 49 = 5 = 5d^2$; then $5 \div 5 = 1 = d^2$, and $\sqrt{1} = 1$; and the four required numbers are, as before, 2, 3, 4 and 5.

2. Given the sum of four numbers in arithmetical progression equal 38, and the sum of their squares equal 406; to find those four numbers.

First $38 \div 4 = 9\frac{1}{2} = m$; then $4m^2 = 361$, and $406 - 361 = 45 = 5d^2 = q$; then $q \div 5 = 9 = d^2$; then $\sqrt{9} = 3 = 2d$; then $d = 1\frac{1}{2}$, and $9\frac{1}{2} - 1\frac{1}{2} = 8 = x$, and $9\frac{1}{2} + 1\frac{1}{2} = 11 = y$; and the four required numbers are 5, 8, 11 and 14.

Or thus: Having found $m = 9\frac{1}{2}$, and $m^2 = 90\frac{1}{4}$, then $406 \div 4 = 101\frac{1}{2}$, and $101\frac{1}{2} - 90\frac{1}{4} = 11\frac{1}{4}$, and $11\frac{1}{4} \div 5 = 2\frac{1}{4} = d^2$, and $\sqrt{2\frac{1}{4}} = 1\frac{1}{2} = d$; and the four required numbers are, as before, 5, 8, 11 and 14.

3. Given the sum of four numbers in arithmetical progression $s = 36$, and the sum of their squares $s^2 = 404$; to find those numbers.

First $s \div n = 9$, and $s^2 \div n = 101 = m^2 + 5d^2$; then $m^2 = 81$, and $101 - 81 = 20 = 5d^2$; then $20 \div 5 = 4 = d^2$, and $\sqrt{4} = 2 = d$; therefore $9 - 2 = 7 = x$, and $9 + 2 = 11 = y$; thus having found the two means and the common difference, the four required numbers will of course be 3, 7, 11 and 15.

Or thus: Having found $m = 9$, then $s^2 - 4m^2 = 80 = 5 \square 2d$; then $80 \div 5 = 16 = \square 2d$; then $\sqrt{16} = 4 = 2d$; and we find the four required numbers, as before, to be 3, 7, 11 and 15.

4. Given the sum of the squares of the two first of four numbers in arithmetical progression equal 65; to find those four numbers.

First $65 \div 2 = 32\frac{1}{2}$; then the number, the square of which is less than $32\frac{1}{2}$, and will leave a difference q equal to a square number, is $5\frac{1}{2}^2 = 30\frac{1}{4}$; then $32\frac{1}{2} - 30\frac{1}{4} = 2\frac{1}{4} = d^2$; then $\sqrt{2\frac{1}{4}} = 1\frac{1}{2} = d$; then $5\frac{1}{2} - 1\frac{1}{2} = 4 = w$, and $5\frac{1}{2} + 1\frac{1}{2} = 7 = x$; thus having found the two first numbers and the common difference, we have of course the four required numbers, 4, 7, 10 and 13.

Or thus: Putting $5\frac{1}{2} = m$, then $2m^2 = 60\frac{1}{2}$, and $65 - 60\frac{1}{2} = 4\frac{1}{2} = 2d^2$, and $4\frac{1}{2} \div 2 = 2\frac{1}{4} = d^2$; and the four required numbers, as before, are 4, 7, 10 and 13.

5. Given the sum of the squares of the two extremes of four numbers in arithmetical progression equal 593; to find those four numbers.

First $593 \div 2 = 296\frac{1}{2} = m^2 + d^2$; then the number, the square of which is less than 296, and leaves a difference q equal to a square number, is $15\frac{1}{2}$; then put $15\frac{1}{2} = m$, and $m^2 = 240\frac{1}{4}$, and $296\frac{1}{2} - 240\frac{1}{4} = 56\frac{1}{4} = d^2$; then $\sqrt{56\frac{1}{4}} = 7\frac{1}{2} = d$; therefore $15\frac{1}{2} - 7\frac{1}{2} = 8 = w$, and $15\frac{1}{2} + 7\frac{1}{2} = 23 = z$; thus having found the two extremes, and their difference equal 15, then $15 \div 3 = 5$, the common difference; and we have the four required numbers, 8, 13, 18 and 23.

7. Given the sum of four numbers in arithmetical progression equal 20, and the difference of the squares of the two extremes equal 60; to find those four numbers.

First $s \div n = 5 = m$, and $60 \div 20 = 3 = d$, or one-half the difference of the two extremes; then $m - d = 2 = w$, and $m + d = 8 = z$; thus having found the two extremes to be 2 and 8, and their difference 6, then $6 \div 3 = 2$, the common difference of the four required numbers; which we find to be 2, 4, 6 and 8.

8. Given the sum of four numbers in arithmetical progression equal 30, and the difference of the squares of the two means equal 45; to find those four numbers.

First $45 \div 30 = 1\frac{1}{2}$, or one-half the difference of the two means; then $30 \div 4 = 7\frac{1}{2} = m$, or a mean proportional; we then have $7\frac{1}{2} - 1\frac{1}{2} = 6 = x$, and $7\frac{1}{2} + 1\frac{1}{2} = 9 = y$; and the four required numbers are thus found to be 3, 6, 9 and 12.

9. Given the sum of four numbers in arithmetical progression equal 30, and the difference of the squares of the two least and the squares of the two greatest equal 180; to find those four numbers. Or thus:

Given $w + x + y + z = 30$, and $(y^2 + z^2) - (w^2 + x^2)$; to find those four numbers.

First $180 \div 30 = 6 = 4d$, or twice the common difference; then $30 \div 4 = 7\frac{1}{2} = m$, or a mean proportional to the four required numbers, and $6 \div 4 = 1\frac{1}{2} = d$, or one-half the common difference; we then have $7\frac{1}{2} - 1\frac{1}{2} = 6 = x$, and $7\frac{1}{2} + 1\frac{1}{2} = 9 = y$; and the four required numbers are 3, 6, 9 and 12.

7. Given the sum of four numbers in arithmetical progression equal 36, and the difference of the squares of the two least and the squares of the two greatest equal 288; to find those four numbers. Or thus:

Given $w+x+y+z=36$, and $(y^2+z^2)-(w^2+x^2)=288$; to find those four numbers.

First $288 \div 36 = 8 = 4d$, or twice the common difference; then $s \div n = m$, or a mean proportional to the four required numbers; then $8 \div 4 = d$, or one-half the common difference; therefore $9-2=7=x$, and $9+2=11=y$; and thus having found the two means and the common difference, the four required numbers will of course be 3, 7, 11 and 15.

CASE IX.

The sum of five numbers in arithmetical progression and the sum of their squares being given; to find those numbers.

RULE.

Let s equal their sum, s^2 the sum of the squares, w the first, x the second, m the third or middle number, y the fourth and z the fifth, n the number of terms, q the difference between $s^2 \div n$ and m^2, and d the common difference; then will $s \div n = m$, and $s^2 \div n = m^2 + 2d^2$, or $s^2 \div n - m^2 = 2d^2$; then $m - d = x$, and $m + d = y$, &c.; or $s^2 = 5m^2 + 10d^2$.

EXAMPLES.

1. Given the sum of five numbers in arithmetical progression $s=30$, and the sum of their squares $s^2=220$; to find those five numbers.

First $s \div n = 6 = m$, and $s^2 \div n = 44 = m^2 + 2d^2$; then $m^2 = 36$, and $44-36=8=2d^2$; then $8 \div 2 = 4 = d^2$, and $\sqrt{4} = 2 = d$; therefore $6-2=4=x$, and $6+2=8=y$; thus having found the three middle numbers and the common difference, we have the five required numbers 2, 4, 6, 8 and 10.

Or thus: Having found $m=6$, then $s^2 - 5m^2 = 10d^2$, and $-5m^2 = 180$; then $220-180=40=10d^2$; then $40 \div 10 = 4 = d^2$, and $\sqrt{4} = 2 = d$; and we find the five required numbers, as before, to be 2, 4, 6, 8 and 10.

2. Given the sum of five numbers in arithmetical progression $s=65$, and the sum of their squares $s^2=1095$; to find those five numbers.

First $65\div5=13=m$, and $1095\div5=219=m^2+2d^2$; then $m^2=169$, and $219-169=50=2d^2$; then $50\div2=25=d^2$, and $\sqrt{25}=5=d$; therefore $13-5=8$, and $13+5=18$; and having thus found the three middle numbers and their common difference, the five required numbers are 3, 8, 13, 18 and 23.

3. Given the sum of the squares of the first and third of five numbers in arithmetical progression equal 180; to find those five numbers.

First $180\div2=90=m^2+d^2$; then the number, the square of which is less than 90, and will leave a difference q equal to a square number, is 9, and $9^2=81$; then $90-81=9=q=d^2$, and $\sqrt{9}=3=d$; therefore $9-3=6=w$, and $9+3=12=m$; and having thus found the three first terms and the common difference, we have of course the five required numbers, 6, 9, 12, 15 and 18.

4. Given the sum of the squares of the two extremes of five numbers in arithmetical progression equal 466; to find those five numbers.

First $466\div2=233=m^2$ (or a mean proportional between the two numbers)$+d^2$; then the number, the square of which is less than 233, and will leave a difference q equal to a square number, is 13; then put $13=m$, and $13^2=169$; then $233-169=64=d^2$, and $\sqrt{64}=8$; therefore $13-8=5$, and $13+8=21$; thus having found the two extremes and their difference equal 16, then $16\div4=4$, the common difference; and the five required numbers are 5, 9, 13, 17 and 21.

Or thus: Putting $13=m$, then $(13-d)^2+(13+d)^2=466$, by the question; i. e. $2m^2+2d^2=466$; then $2m^2=338$, and $466-338=128=2d^2$; then $128\div2=d^2$, or $d=8$; and the five required numbers are, as before, 5, 9, 13, 17 and 21.

5. Given the sum of five numbers in arithmetical presion equal 45, and the difference of the squares of the two extremes equal 216; to find those four numbers.

First $45\div5=9=m$, or a mean proportional to the two extremes, also the mean of the five required numbers; then $216\div m=24=8d$; then $24\div8=3=d$, or the common difference; and the five required numbers are thus found to be 3, 6, 9, 12 and 15.

6. Given the sum of five numbers in arithmetical progression equal 60, and the difference of the squares of the two extremes equal 384; to find those five numbers.

First $60 \div 5$ or $s \div n = 12 = m$; then $384 \div m = 32 = 8d$, and $32 \div 8 = 4 = d$; thus having found $m = 12$, and $d = 4$, the five required numbers are thus found to be 4, 8, 12 16 and 20.

7. Given the sum of five numbers in arithmetical progression equal 45, and the difference of the squares of the two means, or of the second and fourth, equal 108; to find those five numbers.

First $45 \div 5 = 9 = m$; then $108 \div 9 = 12 = 4d$; then $12 \div 4 = 3 = d$; thus having found the mean proportional equal 9, and the common difference equal 3, we have the five required numbers, 3, 6, 9, 12 and 15.

8. Given the sum of five numbers in arithmetical progression equal 85, and the difference of the squares of the two means, or of the second and fourth, equal 340; to find those five numbers.

First $85 \div 5 = 17 = m$; then $340 \div 17 = 20 = 4d$; then $20 \div 4 = 5 = d$; thus having found a mean proportional equal 17, and the common difference equal 5, the five required numbers are thus determined to be 7, 12, 17, 22 and 27.

9. Given the sum of five numbers in arithmetical progression equal 45, and the sum of the squares of one-third of each of them equal 55; to find those five numbers.

First $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$; then $55 \times 9 = 495$, equal the sum of the squares of the five required numbers; then by proceeding as before, we find the five required numbers to be 3, 6, 9, 12 and 15.

CASE X.

The sum of two numbers and the sum of their cubes being given; to find those numbers.

RULE.

Find a mean proportional by dividing the given sum by the number of terms as before; then the sum of the cubes, divided by the number of terms, will be equal to the cube of the mean proportional, more the square of one-half the difference of the two numbers, multiplied by three times the mean proportional. Therefore let x equal the less and y the greater of the two numbers, s their sum, s^3 the sum of their

cubes, n the number of terms, m the mean proportional, q the difference between $s^3 \div n$ and m^3, and $2d$ the difference of the two numbers; then will $s \div n = m$, and $s^3 \div n = m^3 + q$, and $q \div 3m = d^2$; then $m - d = x$, and $m + d = y$.

EXAMPLES.

1. Given the sum of two numbers equal 5, and the sum of their cubes equal 35; to find those two numbers.

First $s \div n = 2\frac{1}{2} = m$, and $s^3 \div n = 17\frac{1}{2} = m^3 + d^2 \times 3m$ or q; then $m^3 = 15\frac{5}{8}$, and $17\frac{1}{2} - 15\frac{5}{8} = 1\frac{7}{8} = q$, and $3m = 7\frac{1}{2}$, and $q \div 7\frac{1}{2} = \frac{1}{4} = d^2$, and $\sqrt{\frac{1}{4}} = \frac{1}{2} = d$; then $2\frac{1}{2} - \frac{1}{2} = 2 = x$, and $2\frac{1}{2} + \frac{1}{2} = 3 = y$; and the two required numbers are 2 and 3.

Or thus: Having found $m = 2\frac{1}{2}$, then $(2\frac{1}{2} - d)^3 + (2\frac{1}{2} + d)^3 = 35$; i. e. $2m^3 + d^2 \times 6m$; then $2m^3 = 31\frac{1}{4}$, and $35 - 31\frac{1}{4} = 3\frac{3}{4}$; then $3\frac{3}{4} \div 6m = \frac{1}{4} = d^2$, and $\sqrt{\frac{1}{4}} = \frac{1}{2} = d$; and we find the two numbers, as before, are 2 and 3.

2. Given the sum of two numbers equal 6, and the sum of their cubes equal 72; to find those two numbers.

First $s \div n = 3 = m$, and $s^3 \div n = 36 = m^3 + q$; then $m^3 = 27$, and $36 - 27 = 9 = q = d^2 \times 3m$; then $9 \div 3m = 1 = d^2$, and $\sqrt{1} = 1 = d$; therefore $3 - 1 = 2 = x$, and $3 + 1 = 4 = y$; and the two required numbers are 2 and 4.

Or thus: Having found $m = 3$, we have by the question $(3-d)^3 + (3+d)^3 = 72$; i. e. $2m^3 + d^2 \times 6m = 72$; then $2m^3 = 18 = q$, and $9 \div 6m = 1 = d^2$; then $3 - 1 = 2 = x$, and $3 + 1 = 4 = y$; and we find the two required numbers, as before, are 2 and 4.

3. Given the sum of two numbers equal 9, and the sum of their cubes equal 243; to find those two numbers.

First $s \div n = 4\frac{1}{2} = m$, and $s^3 \div n = 121\frac{1}{2} = m^3 + q$; then $m^3 = 91\frac{1}{8}$, and $121\frac{1}{2} - 91\frac{1}{8} = 30\frac{3}{8} = q$, and $q \div 3m = 2\frac{1}{4} = d^2$, and $\sqrt{2\frac{1}{4}} = 1\frac{1}{2} = d$; therefore $4\frac{1}{2} - 1\frac{1}{2} = 3 = x$, and $4\frac{1}{2} + 1\frac{1}{2} = 6 = y$; and the two required numbers are 3 and 6.

Or thus: Having found $m = 4\frac{1}{2}$, then $(4\frac{1}{2} - d)^3 + (4\frac{1}{2} + d)^3 = 243$, by the question; then $m^3 = 182\frac{1}{4}$, and $243 - 182\frac{1}{4} = 60\frac{3}{4} = d^2 \times 6m$ or q; then $q \div 6m = 2\frac{1}{4} = d^2$, and $\sqrt{2\frac{1}{4}} = 1\frac{1}{2} = d$; therefore $4\frac{1}{2} - 1\frac{1}{2} = 3 = x$, and $4\frac{1}{2} + 1\frac{1}{2} = 6 = y$; and the two required numbers, as before, are 3 and 6.

4. Given the sum of two numbers equal 60, and the sum of their cubes equal 54720; to find those two numbers.

First $60 \div 2$ or $s \div n = 30 = m$, and $s^3 \div n = 27360 = m^3 + q$; then $m^3 = 27000$, and $27360 - 27000 = 360 = q$; then $q \div 6m = 4 = d^2$, and $\sqrt{4} = 2 = d$; therefore $30 - 2 = 28 = x$, and $30 + 2 = 32 = y$; and the required numbers are 28 and 32.

Or thus: Having found $m=30$, then $(30-d)^3+(30+d)^3=54722$, by the question; i.e. $2m^3+d^2\times 6m$, or equal $2m^3+q$; then $2m^3=54000$, and $54720-54000=720=d^2\times 6m$; then $6m=180$, and $720\div 180=4=d^2$, and $\sqrt{4}=2=d$; therefore $30-2=28=x$, and $30+2=32=y$; and the two required numbers are, as before, 28 and 32.

5. Given the difference of two numbers equal 3, and the difference of their cubes equal 189; to find those numbers.

First $189\div 3$ times $3=21=m^2+\frac{1}{3}d^2$; then the number, the square of which is less than 21, and will leave a difference equal to one-third of a square number, is $4\frac{1}{2}^2=20\frac{1}{4}$; then $21-20\frac{1}{4}=\frac{3}{4}=\frac{1}{3}d^2$; then $\frac{3}{4}\times 3=2\frac{1}{4}=d^2$, and $\sqrt{2\frac{1}{4}}=1\frac{1}{2}=d$; therefore $4\frac{1}{2}-1\frac{1}{2}=3=x$, and $4\frac{1}{2}+1\frac{1}{2}=6=y$; and the two required numbers are 3 and 6.

6. Given the difference of two numbers equal 6, and the difference of their cubes equal 1512; to find those numbers.

Put $2d=6$; then $6d=18$, and $1512\div 18=84=m^2+\frac{1}{3}d^2$; then the number, the square of which is less than 84, and leaves a difference equal to one-third of a square number, is 9; then $9^2=81$, and $84-81=3=\frac{1}{3}d^2$; then $3\times 3=9=d^2$, and $\sqrt{9}=3=d$; therefore $9-3=6=x$, and $9+3=12=y$; and the two required numbers are 6 and 12.

7. Given the difference of two numbers equal 12, and the difference of their cubes equal 7488; to find those numbers.

First $12=2d$; then $6d=36$, and $7488\div 36=208$; then the number, the square of which is less than 288, and, deducted from 288, will leave one-third of a square number, is 14; then $14^2=196$, and $208-196=12=\frac{1}{3}d^2$; then $12\times 3=36=d^2$, and $\sqrt{36}=6=d$; therefore $14-6=8=x$, and $14+6=20=y$; and the two required numbers are 8 and 20.

8. Given the sum of two numbers equal 8, and the difference of their cubes equal 208; to find those numbers.

First $8\div 2=4=m$; then $3m^2=48$, and $208\div 48=4+16=2d^3$; then $16\div 2=8=d^3$, and $\sqrt[3]{8}=2=d$; therefore $4-2=2=x$, and $4+2=6=y$; and the two required numbers are 2 and 6.

9. Given the sum of two numbers equal 12, and the difference of their cubes equal 702; to find those numbers.

First $12\div 2=6=m$, and $3m^2=108$; then $702\div 108=6+54=2d^3$; then $54\div 2=27=d^3$, and $\sqrt[3]{27}=3=d$; therefore $6-3=3=x$, and $6+3=9=y$; and the two required numbers are 3 and 9.

10. Given the sum of two numbers equal 14, and the difference of their cubes equal 1304; to find those numbers.

First $14\div2=7=m$; then $3m^2=147$, and $1304\div147=8+128=2d^3$; then $128\div2=64=d^3$, and $\sqrt[3]{64}=4=d$; therefore $7-4=3=x$, and $7+4=11=y$; and the two required numbers are 3 and 11.

CASE XI.

The sum of three numbers in arithmetical progression and the sum of their cubes being given; to find those three numbers.

RULE.

Find a mean proportional as before, by dividing the given sum of the numbers by the number of terms; also divide the sum of the cubes by the number of terms; then one-third the sum of the cubes, less the cube of the mean proportional, will equal the square of the difference multiplied by $2m$. Or,

Let x equal the least, m the middle or mean proportional, y the greatest, s the sum of the three numbers, s^3 the sum of their cubes, n the number of terms, m the mean proportional, and d the common difference. Then will $s\div n=m$, and $s^3\div n=m^3+d^2\times2m$ or q; therefore $s^3\div n-m^3=q$, and $q\div2m=d^2$; then $m-d=x$, and $m+d=y$; and the three required numbers will be x, m and y.

EXAMPLES.

1. Given the sum of three numbers in arithmetical progression equal 9, and the sum of their cubes equal 99; to find those three numbers.

First $9\div3=$ or $s\div n=3=m$, and $s^3\div n$ or $99\div3=33=m^3+d^2\times2m$; then $m^3=27$, and $33-27=6=d^2\times2m$; then $2m=6$, and $6\div6=1=d^2$, and $\sqrt{1}=1=d$; therefore $3-1=2=x$, $3=m$, and $3+1=4=y$; and the three required numbers are 2, 3 and 4.

Or thus: Having found $m=3$, we have $(3-d)^3+3^3+(3+d)^3=99$, by the question; i. e. $99=3m^3+d^2\times6m$; then $3m^3=81$, and $99-81=18=d^2\times6m$; then $6m=18$, and $18\div18=1=d^2$ and $\sqrt{1}=1=d$; and the three required numbers are, as before, 2, 3 and 4.

2. Given the sum of three numbers in arithmetical progression equal 12, and the sum of their cubes equal 288; to find those three numbers.

First $s \div n = 4 = m$, and $s^3 \div n = 96 = m^3 + d^2 \times 2m$; then $m^3 = 64$, and $96 - 64 = 32 = q$; then $q \div 2m = 4 = d^2$, and $\sqrt{4} = 2 = d$; therefore $4 - 2 = 2 = x$, $4 = m$, and $4 + 2 = 6 = y$; and the three required numbers are 2, 4 and 6.

Or thus: Having found $m = 4$, we have $(4-d)^3 + 4^3 + (4+d)^3 = 288$, by the question; i. e. $288 = 3m^3 + d^2 \times 6m$; then $3m^3 = 192$, and $288 - 192 = 96 = d^2 \times 6m$; then $6m = 24$, and $96 \div 24 = 4 = d^2$, and $\sqrt{4} = 2 = d$; therefore we find the three numbers, as before, to be 2, 4 and 6.

3. Given the sum of three numbers in arithmetical progression equal $37\frac{1}{2}$, and the sum of their cubes equal $6778\frac{1}{8}$; to find those three numbers.

First $37\frac{1}{2} \div 3$ or $s \div n = 12\frac{1}{2} = m$, and $6778\frac{1}{8} \div 3$ or $s^3 \div n = 2259\frac{3}{8}$; then m^3 or $12\frac{1}{2}^3 = 1953\frac{1}{8}$, and $2259\frac{3}{8} - 1953\frac{1}{8} = 306\frac{1}{4}$; then $2m = 55$, and $306\frac{1}{4} \div 25 = 12\frac{1}{4} = d^2$, and $\sqrt{12\frac{1}{4}} = 3\frac{1}{2} = d$; therefore $12\frac{1}{2} - 3\frac{1}{2} = 9 = x$, and $12\frac{1}{2} + 3\frac{1}{2} = 16 = y$; and the three required numbers are 9, $12\frac{1}{2}$ and 16.

Or thus: Having found $m = 12\frac{1}{2}$, then $3m^3 = 5859\frac{3}{8}$, and $s^3 = 6778\frac{1}{8} - 5859\frac{3}{8} = 918\frac{3}{4} = q$, and $6m = 75$; then $q \div 6m = 12\frac{1}{4} = d^2$, and $\sqrt{12\frac{1}{4}} = 3\frac{1}{2} = d$; we then have $12\frac{1}{2} - 3\frac{1}{2} = 9 = x$, and $12\frac{1}{2} + 3\frac{1}{2} = 16 = y$; and the three required numbers are, as before, 9, $12\frac{1}{2}$ and 16.

4. Given the cubes of the two first of three numbers in arithmetical progression equal 793; to find those numbers.

First $793 \div 2 = 396\frac{1}{2}$; then the number, the cube of which is nearest to $396\frac{1}{2}$, but less, and of the same fraction, is $6\frac{1}{2}$; then taking $6\frac{1}{2} = m$ or a mean proportional between those two numbers, we have $6\frac{1}{2}^3$ or $m^3 = 274\frac{5}{8}$; then $396\frac{1}{2} - 274\frac{5}{8} = 121\frac{7}{8}$; then $3m = 19\frac{1}{2}$, and $121\frac{7}{8} \div 19\frac{1}{2} = 6\frac{1}{4} = d^2$; then $\sqrt{6\frac{1}{4}} = 2\frac{1}{2} = d$; therefore $6\frac{1}{2} - 2\frac{1}{2} = 4 = x$, and $6\frac{1}{2} + 2\frac{1}{2} = 9 = m$, or the mean of the three required numbers; and thus having found the two first, we have the three required numbers, 4, 9 and 14.

5. Given the sum of the cubes of the two extremes of three numbers in arithmetical progression equal 36864; to find those three numbers.

First $36864 \div 2 = 18432$; then the number, the cube of which is less than 18432, will easily be found to be 24; then $24^3 = 13844$, and $18432 - 13844 = 4608 = d^2 \times 3m$; then $4608 \div 3m = 64 = d^2$, and $\sqrt{64} = 8 = d$; therefore $24 - 8 = 16 = x$, and $24 + 8 = 32 = y$; thus having found the two extremes of the three numbers to be 16 and 32, and the mean proportional 24, we have the three required numbers equal 16, 24 and 32.

6. Given the difference of the two extremes of three numbers in arithmetical progression equal 6, and the difference of their cubes equal 504; to find those three numbers.

Put $2d$ equal the difference of the two numbers; then $504 \div 6d = m^2 + \frac{1}{3}d^2$, and $6d = 18$, and $504 \div 18 = 28 = m^2 + \frac{1}{3}d^2$; then the number, the square of which is nearest to 28 but less, is 5; then $5^2 = 25$, and $28 - 25 = 3 = \frac{1}{3}d^2$; then $3 \times 3 = 9$, and $\sqrt{9} = 3$; then $5 - 3 = 2 = x$, and $5 + 3 = 8 = y$; and thus having found the mean proportional equal 5, and the common difference equal 3, we find the three required numbers to be 2, 5 and 8.

7. Given the difference of the two first of three numbers in arithmetical progression equal 3, and the difference of their cubes equal 279; to find those three numbers.

Let $2d$ equal the difference as before; then $6d = 9$, and $279 \div 9 = 31$; then the number, the square of which is less than 31, and leaves a difference equal one-third of a square number, is $5\frac{1}{2}$; then $5\frac{1}{2}^2 = 30\frac{1}{4}$, and $31 - 30\frac{1}{4} = \frac{3}{4} = \frac{1}{3}d^2$; then $\frac{3}{4} \times 3 = 2\frac{1}{4} = d^2$, and $\sqrt{2\frac{1}{4}} = 1\frac{1}{2} = d$; therefore $5\frac{1}{2} - 1\frac{1}{2} = 4 = x$, and $5\frac{1}{2} + 1\frac{1}{2} = 7 = m$; thus having found the two first numbers and the common difference, the three required numbers are 4, 7 and 10.

CASE XII.

The sum of four numbers in arithmetical progression and the sum of their cubes being given; to find those numbers.

RULE.

Let w, x, y and z equal the four required numbers, s their sum, s^3 the sum of their cubes, n the number of terms, m the mean proportional, q the difference betweeen $s^3 \div n$ and m^3, and $2d$ the common difference; then will $s \div n = m$, or a mean proportional to the four numbers, $s^3 \div n = m^3 + q$, and $q = 15d^2 \times m$; i. e. $s^3 \div n - m = q$, and $q \div 15m = d^2$; then $m - d = x$, and $m + d = y$, &c.

EXAMPLES.

1. Given the sum of four numbers in arithmetical progression equal 14, and the sum of their cubes equal 224; to find those four numbers.

First $s \div n$ or $14 \div 4 = 3\frac{1}{2} = m$, and $s^3 \div n = 56 = m^3 + d^2 \times 15m$; then $m^3 = 42\frac{7}{8}$, and $56 - 42\frac{7}{8} = 13\frac{1}{8} = q$, and $q \div 15m = \frac{1}{4} = d^2$, and $\sqrt{\frac{1}{4}} = \frac{1}{2} = m$; then $3\frac{1}{2} - \frac{1}{2} = 3 = x$, and $3\frac{1}{2} + \frac{1}{2} = 4 = y$; and the four required numbers are 2, 3, 4 and 5.

Or thus: $s^3-4m^3\div 15m=d^2$; thus in the above example the cube of m is found equal $42\frac{7}{8}$, and $4m^3=171\frac{1}{2}$; then $224-171\frac{1}{2}=52\frac{1}{2}$, and $15m=52\frac{1}{2}$; then $52\frac{1}{2}\div 52\frac{1}{2}=1=d^2$, and $\sqrt{1}=1=d$; and the four required numbers are, as before, 2, 3, 4 and 5.

2. Given the sum of four numbers in arithmetical progression equal 20, and the sum of their cubes equal 800; to find those four numbers.

First $s\div n=5=m$, and $s^3\div n=200=m^3+15md^2$; then $m^3=125$, and $200-125=75$, and $15m=75$; then $75\div 75=1=d^2$, and $\sqrt{1}=1=d$; therefore $5-1=4=x$, and $5+1=6=y$; and the four required numbers are 2, 4, 6 and 8.

Or thus: Having found $m^3=125$, then $4m^3=500$, and $800-500=300=q$; then $15m=75$, and $300\div 75=4=\square\, 2d$, and $\sqrt{4}=2=2d$; and we find the four required numbers to be, as before, 2, 4, 6 and 8.

3. Given the sum of four numbers in arithmetical progression equal 30, and the sum of their cubes equal 2700; to find those four numbers.

First $30\div 4$ or $s\div n=7\frac{1}{2}=m$, and $2700\div 4$ or $s^3\div n=675$; then $m^3=421\frac{7}{8}$, and $675-421\frac{7}{8}=253\frac{1}{8}=q$, and $15m=112\frac{1}{2}$; then $253\frac{1}{8}\div 112\frac{1}{2}=2\frac{1}{4}=d^2$, and $\sqrt{2\frac{1}{4}}=1\frac{1}{2}=d$; therefore $7\frac{1}{2}-1\frac{1}{2}=6=x$, and $7\frac{1}{2}+1\frac{1}{2}=9=y$; and the four required numbers are 3, 6, 9 and 12.

Or thus: Having found $m^3=421\frac{7}{8}$, then $4m^3=1687\frac{1}{2}$, and $2700-1687\frac{1}{2}=1012\frac{1}{2}$; then $15m=112\frac{1}{2}$, and $1012\frac{1}{2}\div 112\frac{1}{2}=9=\square\, 2d$, and $\sqrt{9}=3=2d$; and the four required numbers, as before, are 3, 6, 9 and 12.

4. Given the sum of the cubes of the two first of four numbers in arithmetical progression equal 854; to find those four numbers.

First $854\div 2=427$; then the number, the cube of which is nearest to 427, but less, will easily be found to be 7; then taking $7=m$ or a mean proportional between those two numbers, we have 7^3 or $m^3=343$; then $427-423=84=q=d^2\times 3m$; then $3m=21$, and $84\div 21=4=d^2$, and $\sqrt{4}=2=d$; therefore $7-2=5=w$, and $7+2=9=x$; and thus having found the two first to be 5 and 9, we have the four required numbers, 5, 9, 13 and 17.

5. Given the sum of the cubes of the first and third of four numbers in arithmetical progression equal 2224; to find those four numbers.

First $2224\div 2=1112=m^3+3md^2$; then the number, the cube of which is less than 1112, and will answer the con-

ditions of the rule, will be found to be $8=m$; then $m^3=512$, and $1112-512=600=q$; then $3m=24$, and $600\div24=25=d^2$, and $\sqrt{25}=5=d$; therefore $8-5=3=w$, $8=x$, and $8+5=13=y$; and having thus found the three first to be 3, 8 and 13, we have of course the four required numbers, 3, 8, 13 and 18.

Or thus: Putting $8=m$ or $=x$, then $2m^3=1024$, and $2224-1024=1200=3m\times2d^2$; then $3m=24$, and $1200\div24=50=2d^2$; then $50\div2=25=d^2$, and $\sqrt{25}=5=d$; and the four required numbers, as before, are 3, 8, 13 and 18.

6. Given the difference of the two extremes of four numbers in arithmetical progression equal 12, and the difference of their cubes equal 3348; to find those four numbers.

First $12\times3=36$; then $3344\div36=93=m^2+\frac{1}{3}d^2$, and $\sqrt{93}=9+12=\frac{1}{3}d^2$; then $12\times3=36=d^2$, and $\sqrt{36}=6=d$; therefore $9-6=3=w$, and $9+6=15=z$; thus having found the two extremes and the common difference, the four required numbers are 3, 7, 11 and 15.

7. Given the difference of the two means of four numbers in arithmetical progression equal 4, and the difference of their cubes equal 988; to find those four numbers.

First $4\times3=12$, and $988\div12=82\frac{1}{3}=m^2+\frac{1}{3}d^2$; then $\sqrt{82\frac{1}{3}}=9+1\frac{1}{3}=\frac{1}{3}d^2$; then $1\frac{1}{3}\times3=4=d^2$, and $\sqrt{4}=2=d$; therefore $9-2=7=x$, and $9+2=11=y$; and having thus found the two middle numbers and their common difference, the four required numbers are 3, 7, 11 and 15, as per last.

CASE XIII.

The sum of five numbers in arithmetical progression and the sum of their cubes being given; to find those numbers.

RULE.

Let s equal their sum, s^3 the sum of their cubes, w the first, x the second, m the third or middle number, y the fourth and z the fifth, n the number of terms, q the difference between m^3 and $\frac{1}{5}$ of s^3, and d the common difference; then will $s\div n=m$, and $s^3\div n=m^3+d^2\times6m$; i. e. $s^3\div n-m^3=q$, and $q\div6m=d^2$; then $m-d=x$, and $m+d=y$, &c.

EXAMPLES.

1. Given the sum of five numbers in arithmetical progression equal 35, and the sum of their cubes equal 2555; to find those five numbers.

First $s\div n=7=m$, and $s^3\div n=551=m^3+d^2\times6m$; then

$m^3=343$, and $551-343=168=q$; then $6m=42$, and $q\div 42=4=d^2$, and $\sqrt{4}=2=d$; therefore $7-2=5=x$, and $7+2=9=y$; thus having found the three middle numbers and the common difference, we have the five required numbers 3, 5, 7, 9 and 11.

Or thus: Having found $m=7$, then $s^3-5m^3=30md^2$, and $5m^3=1715$; then $2555-1715=810$, and $30m=210$; then $840\div 210=4=d^2$, and $\sqrt{4}=2=d$; and the five required numbers are, as before, 3, 5, 7, 9 and 11.

2. Given the sum of five numbers in arithmetical progression equal 6, and the sum of their cubes equal 11880; to find those five numbers.

First $s\div n=12=m$, and $s^3\div n=2376=m^3+6md^2$; then $m^3=1728$, and $2376-1728=648=q$; then $6m=72$, and $q\div 72=9=d^2$, and $\sqrt{9}=3=d$; therefore $12=m$, $12-3=9=x$, and $12+3=15=y$; and the five required numbers are 6, 9, 12, 15 and 18.

Or thus: Having found $5m^3=8640$, we have $11880-8640=3240=30md^2$, and $30m=360$; then $3240\div 360=9=d^2$, and $\sqrt{9}=3=d$; we then have $12-3=9=x$, and $12+3=15=y$; and the five required numbers are, as before, 6, 9, 12, 15 and 18.

3. Given the sum of five numbers in arithmetical progression equal 85, and the sum of their cubes equal 37315; to find those five numbers.

First $s\div n=17=m$, and $s^3\div n=7463=m^3+6md^2$; then $m^3=4913$, and $7463-4613=2550=6md^2$; then $6m=102$, and $2550\div 102=25=d^2$, and $\sqrt{25}=5=d$; then $17-5=12=x$, and $17+5=22=y$; and thus having found the three means, the five required numbers are 7, 12, 17, 22 and 27.

Or thus: Put $5m^3=24565$, and $37315-24565=12750=30md^2$; then $30m=510$, and $12750\div 510=25=d^2$, and $\sqrt{25}=5=d$; therefore we find the five numbers, as before, to be 7, 12, 17, 22 and 27.

4. Given the sum of the cubes of the three first of five numbers in arithmetical progression equal 3540; to find those five numbers.

First $3540\div 3=1180$; then the number, the cube of which is less than 1180, will easily be found to be 10; then $10^3=1000$, and $1180-1000=180=q$; then $q\div 2m=9=d^2$, and $\sqrt{9}=3=d$; therefore $10-3=7=w$, $x=10$, and $10+3=13=m$; thus having found the two first of the five numbers to be 7 and 10, and the mean proportional 13, we have the five required numbers equal 7, 10, 13, 16 and 19.

Or thus: Having found the middle number of the three equal 10, then $3m^3=3000$, and $3540-3000=540=6md^2$; then $6m=60$, and $540\div 60=9=d^2$, and $\sqrt{9}=3=d$; and the five required numbers are, as before, 7, 10, 13, 16 and 19.

5. Given the sum of the cubes of the two extremes of five numbers in arithmetical progression equal 12194; to find those five numbers.

First $12194\div 2=6097$; then the number, the cube of which is nearest to 6097, and agrees with the conditions of the rule, will be found to be 13, and $13=m$; then $13^3=2197$, and $6097-2197=3900=q$; then $3m=39$, and $q\div 39=100=d^2$, and $\sqrt{100}=10=d$; then $13-10=3=w$, and $13+10=23=z$; and thus having found the mean proportional equal 13, and the difference of the two extremes equal 20, the common difference is of course 5, and we find the five required numbers to be 3, 8, 13, 18 and 23.

6. Given the sum of the cubes of the two means (or of the second and fourth) of five numbers in arithmetical progression equal 6344; to find those five numbers.

First find by trial a number, the cube of which is less than $6344\div 2$, or less than 3172, and the remainder of which, divided by $3m$, will be a square number; thus $13^3=2197$, and then $3172-2197=975$, and $3m=39$; then $975\div 39=25=d^2$, and $\sqrt{25}=5=d$; therefore $13-5=8$, and $13+5=18$; thus we find $x=8$, $m=13$, and $y=18$; and having found these three numbers and their common difference, the five required numbers are 3, 8, 13, 18 and 23.

7. Given the difference of the two extremes of three numbers in arithmetical progression equal 12, and the difference of their cubes equal 2736; to find those five numbers.

First $12\times 3=36$, and $2736\div 36=76=m^2+\frac{1}{3}d^2$; then $\sqrt{76}=8+12=\frac{1}{3}d^2$, and $12\times 3=36=d^2$; then $\sqrt{36}=6=d$; therefore $8-6=2=w$, and $8+6=14=z$; and thus having found the two extremes, the five required numbers will of course be 2, 5, 8, 11 and 14.

CASE XIV.

The sum of two numbers and the sum of their fourth powers being given; to find those numbers.

RULE.

Let x equal the less and y the greater of the two numbers, s their sum, s^4 the sum of their fourth powers, n the number of terms, m the mean proportional, q the difference between

$2m^4$ and s^4, and $2d$ the difference of the two numbers; then will $s^4-2m^4=q=12m^2\times d^2+d^4$, and $q\div 12m^2=d^2+2d^4$; then $m-d=x$, and $m+d=y$; or $(m-d)^4+(m+d)^4-2m^4+12m^2\times 2d^2+2d^4$.

EXAMPLES.

1. Given the sum of two numbers equal 6, and the sum of their 4th powers equal 272; to find those two numbers.

First $s\div n=3=m$, and $(3-d)^4+(3+d)^4=272$, by the question; then $2m^4=162$, and $272-162=110=12m^2\times d^2+2d^4$; then $12m^2=108$, and $110\div 108=1+2=2d^4$; then $2\div 2=1=d^4$, and $\sqrt[4]{1}=1=d$; then $3-1=2=x$, and $3+1=4=y$; and the two required numbers are 2 and 4.

2. Given the sum of two numbers equal 8, and the sum of their 4th powers equal 1312; to find those two numbers.

First $s\div n=4=m$, and $s^4\div n=656$; then $m^4=256$, and $656-256=400=q=12m^2\times d^4+d^4$; then $12m^2=192$, and $400\div 192=2+16=\frac{1}{2}d^2$, and $\sqrt[4]{16}=2=d$; therefore $4-2=2=x$, and $4+2=6=y$; and the two required numbers are 2 and 6.

Or thus: Having found $m=4$, we have by the question $(4-d)^4+(4+d)^4=1312$; then $2m^4=512$, and $1312-512=800$; then $12m^2=192$, and $800\div 192=4=d^2$, leaving a remainder equal $32=2d^4$, and $32\div 2=16=d^4$; then $\sqrt[4]{16}=2=d$; therefore $4-2=2=x$, and $4+2=6=y$; and we find the two required numbers, as before, are 2 and 6.

3. Given the sum of two numbers equal 12, and the sum of their 4th powers equal 6642; to find those two numbers.

First $12\div 2$ or $s\div n=6=m$, and $s^4\div n=3321$; then $m^4=1296$, and $3321-1296=2025=q$; then $12m^2=432$, and $2025\div 432=4\frac{1}{2}$, or $4\frac{1}{2}=\frac{1}{2}d^2$, leaving a remainder of $81=d^4$, and $\sqrt[4]{81}=3=d$; therefore $6-3=3=x$, and $6+3=9=y$; and the two required numbers are 3 and 9.

Or thus: Having found $m=6$, then $(6-d)^4+(6+d)^4=6642$, by the question; then $2m^4=2592$, and $6642-2592=4050$; then $12m^2=432$, and $4050\div 432=9=d^2$, leaving a remainder of $162=2d^4$; then $162\div 2=81=d^4$, and $\sqrt[4]{81}=3=d$; therefore $6-3=3=x$, and $6+3=9=y$; and the two required numbers are, as before, 3 and 9.

4. Given the sum of two numbers equal 8, and the difference of their 4th powers equal 1280; to find those two numbers.

First $s\div n=4=m$; we then have $(4+d)^4-(4-d)^4=$

1280, by the question; then $2m^4=512$, and $1280-512=768=12m^2\times d^2$, when m and d^2 are equal; when they are not, the difference will equal x; then $12m^2=192$, and $768\div192=4=d^2$, and no remainder, which shows d^2 and x^2 are equal; then $\sqrt{4}=2=d$ or x; therefore $4-2=2=x$, and as the two numbers are given equal 8, then $8-2=6=y$; and the two required numbers are 2 and 6.

5. Given the sum of two numbers equal 14, and the difference of their 4th powers equal 14560; to find those two numbers.

First $14\div2=7=m$; then, as before, $(7+d)^4-(7-d)^4=14560$, by the question; then $2m^4=4802$, and $14560-4802=9758=12m^2d^2+2d^4-2x^4$; then $12m^2=588$, and $9758\div588=16=d^2$, leaving a remainder of $350=2d^4-x^4$; then $350\div2=175=d^4-x^4$; and as the nearest fourth power that is greater than 175 is $4^4=256$, then $256-175=81$ x^4, and $\sqrt[4]{81}=3=x$; then having the sum of the two numbers given equal 14, then $14-3=11=y$; and the two required numbers are 3 and 11.

6. Given the difference of two numbers equal 2, and the difference of their 4th powers equal 240; to find those two numbers.

First $2\times4=8$; then $240\div8=30=m^3+md^2$; then the number, the cube of which is less than 30, is 3, and $3^3=27$; then $30-27=3=md^2$, and $3\div3=1=d^2$, and $\sqrt{1}=1=d$; therefore $3-1=2=x$, $3=m$, and $3+1=4=y$; and the two required numbers are 2 and 4.

7. Given the difference of two numbers equal 8, and the difference of their 4th powers equal 14560; to find those two numbers.*

First $8\times4=32=8d$, putting $8=2d$; then $14560\div32=445=m^3+md^2$; then the number, the cube of which is less than 455, is 7; then $7=m$, and $m^3=343$; then $455-343=112=md^2$; then $112\div m=16=d^2$, and $\sqrt{16}=4=d$; therefore $7-4=3=x$, and $7+4=11=y$; and the two required numbers are 3 and 11.

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*This is given in Mr. Ryan's edition of Bonnycastle's Algebra, New-York, 1822, page 145, where he observes, "In the solution of this question, the process brings out the answer in the form $x^3+ax=b$, which is a cubic equation, and therefore cannot be resolved by the ordinary rules of quadratics, but we can *sometimes* reduce such equations to the form of quadratic, and then resolve them by the rules already given:" by which he appears to have no certain or general rule. The rule here given is certain, general and simple.

8. Given the difference of two numbers equal 7, and the difference of their fourth powers equal 20111; to find those two numbers.

First $7\times4=28=8d$; then $20111\div28=718\frac{1}{4}$; then by extracting the nearest cube root, we find $8\frac{1}{2}^3=614\frac{1}{8}$; then $718\frac{1}{4}-614\frac{1}{8}=104\frac{1}{8}\div m=12\frac{1}{4}$, which brings out $d^2=12\frac{1}{4}$; then $\sqrt{12\frac{1}{4}}=3\frac{1}{2}=d$; therefore $8\frac{1}{2}-3\frac{1}{2}=5=x$, and $8\frac{1}{2}+3\frac{1}{2}=12=y$; and the two required numbers are 5 and 12.

9. Given the difference of two numbers equal 3, and the difference of their 4th powers equal 1215; to find those two numbers.

First $3\times4=12=8d$, and $1215\div12=101\frac{1}{4}=m^3+md^2$; then the number, the cube of which is less than $101\frac{1}{4}$, is $4\frac{1}{2}$, and $4\frac{1}{2}^3=91\frac{1}{8}$; then $101\frac{1}{4}-91\frac{1}{8}=10\frac{1}{8}=md^2$; then $10\frac{1}{8}\div4\frac{1}{2}=2\frac{1}{4}=d^2$, and $\sqrt{2\frac{1}{4}}=1\frac{1}{2}=d$; we then have $4\frac{1}{2}-1\frac{1}{2}=3=x$, and $4\frac{1}{2}+1\frac{1}{2}=6=y$; and the two required numbers are 3 and 6.

CASE XV.

The sum of three numbers in arithmetical progression and the sum of their 4th powers being given; to find those three numbers.

RULE.

Find a mean proportional as before, by dividing the given sum of the numbers by the number of terms; then from the sum of the 4th power deduct $3m^4$, and the remainder, equal q, divided by $12m^2$, will leave a remainder equal twice the 4th power of the common difference. Or thus:

Let x equal the least, m the middle or mean proportional, y the greatest, s the sum of the three numbers, s^4 the sum of their 4th powers, n the number of terms, q the difference between s^4 and $3m^4$, and d the common difference. Then will $s^4-3m^4=12m^2d^4$; i. e. $s^4-3m^4\div12m^2=d^2+2d^4$, or will leave a remainder equal $2d^4$.

EXAMPLES.

1. Given the sum of three numbers in arithmetical progression equal 12, and the sum of their 4th powers equal 1568; to find those three numbers.

First $12\div3=4=m$; we have $(m-d)^4+m^4+(m+d)^4=1568$, by the question; then $3m^4=768$, and $1568-768=800=q=12m^2d^2+2d_4$; then $12m^2=192$, and $800\div192=4+32=2d^4$; then $32\div2=16=d^4$, and $\sqrt[4]{16}=2=d$;

therefore $4=m$, $4-2=2=x$, and $4+2=6=y$; and the three required numbers are 2, 4 and 6.

2. Given the sum of three numbers in arithmetical progression equal 15, and the sum of their 4th powers equal 4737; to find those three numbers.

First $15\div3$ or $s\div n=5=m$; we then have $(5-d)^4+5^4+(5+d)^4=4737$, by the question; then $3m^4=1875$, and $4737-1875=2862=12m^2d^2+2d^4$; then $12m^2=300$, and $2862\div300\div9=d^2+162=2d^4$; then $162\div2=81=d^4$, and $\sqrt[4]{81}=3=d$; therefore $5-3=2=x$, $5=m$, and $5+3=8=y$; and the three required numbers are 2, 5 and 8.

3. Given the sum of three numbers in arithmetical progression equal 24, and the sum of their 4th powers equal 32738; to find those three numbers.

First $24\div3=8=m$; then $3m^4$ 12283, and $32738-12288=20450=q$; then $12m^2=768$, and $20450\div768=25+1250=2d^4$; then $1250\div2=625=d^4$, and $\sqrt[4]{625}=5=d$; therefore $8-5=3=x$, $8=m$, and $8+5=13=y$; and the three required numbers are 3, 8 and 13.

4. Given the difference of the two extremes of three numbers in arithmetical progression equal 8, and the difference of their 4th powers equal 27936; to find those three numbers.

This may be solved the same as if there were only two given numbers, taking the middle number for a mean proportional. Thus, put $8=2d$; then $8\times4=32=8d$; then $27936\div32=873$, leaving a remainder equal 144; then the number, the cube of which is nearest to 837, but less, is 9; and putting $9=m$, then $m^3=729$, and $873-729=144=md^2$; then $144\div9=16=d^2$, and $\sqrt{16}=4=d$; therefore $9-4=5=x$, $9=m$, and $9+4=13=y$; and the three required numbers are 5, 9 and 13.

CASE XVI.

The sum of two numbers and the sum of their 5th powers being given; to find those numbers.

RULE.

Find a mean proportional by dividing the given sum by the number of terms as before; then from half the sum of the 5th power of the two numbers deduct the 5th power of the mean proportional; the remainder will be equal to the mean proportional multiplied by ten times the square of one

half the difference, more five times the 4th power of one half the difference; therefore the remainder, or q, divided by ten times the square of the mean proportional multiplied by the square of one half the difference, will give a quotient equal to the mean proportional, and leave a remainder equal to the 4th power of one-half the difference multiplied by five times the mean proportional. Then to find the square of one-half the difference, proceed thus: Divide q by ten times the square of m, and divide that quotient by one-half of s, so as to have a quotient that is a square number; thus, suppose q equal to 128160, which it will be when the required numbers are 5 and 13; then $m^2=81$, and $10m^2=810$; then $128160\div810=158$, and a remainder of 180; then $158\div9=17$, and a remainder of 5; but 17 is not a square number; therefore take 16, and $810\times16=12960$; then $128260\div12960=9$, and a remainder of 11520; then $11520\div5m$ or $45=256=$4th power of d, or one-half the difference; then $\sqrt[4]{256}=4=d$; therefore $9-4=5=x$, and $9+4=13=y$; and the two required numbers, as before, are 5 and 13.

This rule, I am sensible, will appear at first somewhat abstruse; but a little practice and a competent teacher will soon make it easy.

EXAMPLES.

1. Given the sum of two numbers equal 6, and the sum of their 5th powers equal 1056; to find those two numbers.

First $s\div n=3=m$, and $s^5\div n=528$; then $m^5=243$, and $528-243=285$; then $10m^2=90$, and $285\div90=3$, leaving a remainder of 15; then $3\div3=1=d^2$, or $3\div\frac{1}{2}s=3=1$; then $15\div5m$ or $15=1=d^2$, and $\sqrt{1}=1=d$; therefore $3-1=2=x$, and $3+1=4=y$; and the two required numbers are 2 and 4.

2. Given the sum of two numbers equal 8, and the sum of their 5th powers equal 7808; to find those two numbers.

First $s\div n=4=m$, and $s^5\div n=3904$; then $m^5=1024$, and $3904-1024=2880=q$; then $10m^2=160$, and $2880\div160=18$; then $18\div(\frac{1}{2}s\div n)=4+2$, and as 4 is a square number equal d^2, then $160\times4=640$, and $2880\div640=4$, leaving a remainder equal 320; then $5m=20$, and $320\div20=16=d^4$, and $\sqrt[4]{16}=2=d$; therefore $4-2=2=x$, and $4+2=6=y$; and the two required numbers are 2 and 6.

3. Given the sum of two numbers equal 12, and the sum of their 5th powers equal 59292; to find those two numbers.

First $12\div 2$ or $s\div n=6=m$, and $s^5\div n=29646$; then $m^5=7776$, and $29646-7776=21870=q$; then $10m^2=360$, and $21870\div 360=60$; then $60\div\frac{1}{2}s$ or $6=10$, which is not a square number; then take $9=d^2$, and $360\times 9=3240$; then $21870\div 3240=6=m$, leaving a remainder equal 2430; then $5m=30$, and $2430\div 30=81=d^4$, and $\sqrt[4]{81}=3=d$; therefore $6-3=3=x$, and $6+3=9=y$; and the two required numbers are 3 and 9.

4. Given the sum of two numbers equal 16, and the sum of their 5th powers equal 249856; to find those numbers.

First $s\div n=8=m$, and $s^5\div n=124978$; then $m^5=32768$, and $124978-32768=92210$; then $10m^2=640$, and $92110\div 640=143$, leaving a remainder; then $143\div 8=17$, and a remainder; but 17 is not a square number; then take 16, and $640\times 16=10240$; then $92210\div 10240=8$, leaving a remainder of 10240; then $5m=40$, and $10240\div 40=256=d^4$, and $\sqrt[4]{256}=4=d$; therefore $8-4=4=x$, and $8+4=12=y$; and the two required numbers are 4 and 12.

Or these problems may be solved thus:

$s^5-2m^5\div 20m^2d^2=d^4$; and the square of d may be found thus: $s^5-2m^5=q$; then $q\div 20m^2$, and that quotient, divided by m, so as to produce a square number, will be equal to d^2; thus having found d^2 and the mean proportional, the two numbers may be found without further process. But I think it best to proceed thus: $q\div 10m^2d^2=m$, and leaves a remainder equal $d^4\times 10m$; therefore that remainder, divided by $10m$, will equal d^4.

EXAMPLES.

5. Given the sum of two numbers equal 8, and the sum of their 5th powers equal 7808; to find those numbers.

First $8\div 2=4=m$; then $(4-d)^5+(4+d)^5=7808$, by the question; then $2m^5=2048$, and $7808-2048=5760$; then $5760\div 20m^2=18$, and $18\div m=4+2$, and 4 is a square number equal d^2; then $20m^2d^2=1280$, and $5760\div 1280=4=m$, leaving a remainder equal $640=10md^4$; then $10m=40$, and $640\div 40=16=d^4$, and $\sqrt[4]{16}=2=d$; therefore $4-2=2=x$, and $4+2=6=y$; and the two required numbers are 2 and 6.

6. Given the sum of two numbers equal 10, and the sum of their 5th powers equal 17050; to find those numbers.

First $s\div n=5=m$; then $2m^5=6250$, and $17050-6250=10800=q$; then $20m^2=500$, and $q\div 20m^2=21$; then $21\div m=4$, leaving a remainder, and 4 is a square number

equal d^2; then $20m^2d^2=2000$, and $q\div 20m^2d^2=5=m$, leaving a remainder of $800=10md^4$; then $800\div 10m=16=$ $=d^4$; therefore $5-2=3=x$, and $5+2=7=y$; and the two required numbers are 3 and 7.

7. Given the sum of two numbers equal 16, and the sum of their 5th powers equal 249856; to find those numbers.

First $s\div n=8=m$, and $2m^5=65536$; then $s^5-2m^5=$ $184320=q$, and $20m^2=1280$; then $q\div 20m^2=144$, and 144 $\div m=18$, which is not a square number; then take $16=d^2$, and $20m^2d^2=20480$; then $q\div 20m^2d^2=8=m$, leaving a remainder equal $20450=10md^4$; then $20480\div 10m=256$ $=d^4$, and $\sqrt[4]{256}=4=d$; we then have $8-4=4=x$, and $8+4=12=y$; and the two required numbers are 4 and 12.

8. Given the difference of two numbers equal 4, and the difference of their 5th powers equal 7744; to find those two numbers.

First let the difference $4=2d$, and $10d=20$; then 7744 $\div 20=387.2=m^4+\frac{1}{5}10mm^2+d^4$; then the number, the 4th power of which is less than 387.2, is $4^4=256$; then let 4 $=m$, and $387.2-256=131.2\times 5=656=10mm^2+d^4$; then $10m^2=160$, and $656\div 160=4$, leaving a remainder of 16 $=d^4$, and $\sqrt[4]{16}=2=d$; therefore $4-2=2=x$, and $4+2=$ $6=y$; and the two required numbers are 2 and 6.

9. Given the difference of two numbers equal 5, and the difference of their 5th powers equal 96875; to find those two numbers.

First $5=2d$; then $10d=25$, and $96875\div 25=3875=m^4$ $+\frac{1}{5}10m^2+d^4$; then the number, the 4th power of which is less than 3875, is $7\frac{1}{2}^4=3164\frac{1}{16}$, and $3875-3164\frac{1}{16}=710\frac{15}{16}$; and $710\frac{15}{16}\times 5=3554\frac{11}{16}$; then m^2 or $7\frac{1}{2}^2=56\frac{1}{4}$, and $10m^2$ $=562\frac{1}{2}$; then $3554\frac{11}{16}\div 562\frac{1}{2}=6\frac{3}{4}\frac{?}{}$, leaving a remainder of $39\frac{1}{16}=d^4$, and $\sqrt[4]{39\frac{1}{16}}=2\frac{1}{2}=d$; therefore $7\frac{1}{2}-2\frac{1}{2}=5=x$, and $7\frac{1}{2}+2\frac{1}{2}=10=y$; and the two required numbers are 5 and 10.

CASE XVII.

The sum of three numbers in arithmetical progression and the sum of their 5th powers being given; to find those three numbers.

RULE.

Find a mean proportional as before, by dividing the given sum of the numbers by the number of terms; then from the sum of their 5th powers deduct the 5th power of the

mean proportional, and the question will stand the same as if two numbers and their 5th powers were given, the mean proportional being the middle number. Thus:

Given the sum of three numbers in arithmetical progression equal 24, and the sum of their 5th powers equal 282624; to find those three numbers.

First $24\div3=8=m$, the middle number; then $8^5=32768$ and $282624-32768=249856$, the sum of the 5th power of two numbers whose sum is 16; as in case xvi. example 4th; and the three required numbers are 4, 8 and 12.

Other rules may be given for the solution of problems of this nature; but I think this so much easier than any other, that it is unnecessary to insert them.

CASE XVIII.

The sum of three numbers in geometrical progression and the sum of their squares being given; to find those numbers.

RULE.

In any three numbers in geometrical progression the sum of the squares, divided by the ratio more 1, is equal to the square of the second term, more the square of the first. Thus when the ratio is given, this is the most easy method of solution; but when the ratio is not given, let it be noted that the square of the middle term is equal to the product of the two extremes: therefore let x, y and z equal the three required numbers; then $y^2+xz=2y^2$: or the sum of the squares, divided by the number of terms, is equal to the square of a mean proportional to the two extremes, more one-third of their difference; and thus having found the two extremes, their product is equal to the square of the second term.

EXAMPLES.

1. Given the sum of three numbers in geometrical progression equal 14, and the sum of their squares equal 84; to find those three numbers; the ratio being 2.

First $r^2+1=5$, and $84\div5=16+4$; then $\sqrt{16}=4=y$, and $\sqrt{4}=2=x$; thus having found the two first numbers to be 2 and 4, the third will of course be 8; and the three required numbers are 2, 4 and 8.

Or if the ratio were not given, then proceed thus: $84\div3=28=m^2+\frac{1}{3}d^2$, or one-third the square of one-half the difference of the two extremes; then the number, the square of

which is less than 28, and will leave a remainder equal one-third of a square, is $5^2=25$; then $28-25=3=\frac{1}{3}d^2$, and $3\times 3=9$, and $\sqrt{9}=3=d$; therefore $5-3=2=x$, and $5+3=8=z$; and thus having found the two extremes to be 2 and 8, then $8\times 2=16=y^2$, and $\sqrt{16}=4=y$; and the three required numbers are 2, 4 and 8.

2. Given the sum of three numbers in geometrical progression equal 39, and the sum of their squares equal 819; to find those three numbers; the ratio being 3.

First $r^2+1=10$, and $819\div 10=81+9$; then $\sqrt{81}=9=y$, and $\sqrt{9}=3=x$; thus having found the two first numbers to be 3 and 9, we have of course the three required numbers, 3, 9 and 27.

Or if no ratio were given, proceed thus: $819\div 3=273$; then the number, the square of which is less than 273, and will leave a remainder equal to one-third of a square number, is $15^2=225$; then $273-225=48=\frac{1}{3}d^2$, and $48\times 3=144=d^2$, and $\sqrt{144}=12=d$; therefore $15-12=3=x$, and $15+12=27=z$; and having thus found x and z, and the product of $xz=81=y^2$, then $\sqrt{81}=9=y$; and the three required numbers, as before, are 3, 9 and 27.

3. Given the sum of three numbers in geometrical progression equal 26, and the sum of their squares equal 364; to find those three numbers; the ratio being 3.

First $r^2+1=10$, and $364\div 10=36+4$; then $\sqrt{36}=6=y$, and $\sqrt{4}=2=x$; and having found the two first to be 2 and 6, the three required numbers are 2, 6 and 18.

Or if no ratio were given, proceed thus: $364\div 3=121\frac{1}{3}=m^2+\frac{1}{3}d^2$; then the number, the square of which is less than $121\frac{1}{3}$, leaving a remainder equal one-third of a square number, is $10^2=100$; then $121\frac{1}{3}-100=21\frac{1}{3}$, and $21\frac{1}{3}\times 3=64=d^2$, and $\sqrt{64}=8=d$; therefore $10-8=2=x$, and $10+8=18=z$, and $xz=36=y^2$; then $\sqrt{36}=6=y$; and the three required numbers, as before, are 2, 6 and 18.

NOTE.—When the sum of the squares form a fraction, divide, so that the quotient may be a square number. A little attention and practice will make this easy. Thus:

4. Given the sum of three numbers in geometrical progression equal $24\frac{3}{8}$, and the sum of their squares equal 289.453125; to find those numbers; the ratio being $2\frac{1}{2}$.

First $r^2+1=7.25$, and $289.453125\div 7.25=39$, the whole number; then the only square number between 39 and 40 is $39\frac{1}{16}$ or 39.0625, and $39.0625\times 7.25=283.203125$; then $289.453125-283.203125=6.25=x^2$, and $\sqrt{6.25}=2.5=x$,

and $\sqrt{39\frac{1}{16}}=6\frac{1}{4}=y$; and thus having found the two first numbers equal $2\frac{1}{2}$ and $6\frac{1}{4}$, the three required numbers are of course $2\frac{1}{2}$, $6\frac{1}{4}$ and $15\frac{5}{8}$.

Or proceed thus, when no ratio is given: $289.453125 \div 3 = 96.484375$; then the number, the square of which is nearest to this, but less, is $9\frac{1}{16}^2=82.12890625$, and $96.484375-82.12890625=14.35546875=\frac{1}{3}d^2$; this multiplied by 3, and extracting the root as before given, will leave $d=6\frac{9}{16}$; then m or $9\frac{1}{16}-6\frac{9}{16}=x$, and $9\frac{1}{16}+6\frac{9}{16}=15\frac{5}{8}$; thus having found the two first equal $2\frac{1}{2}$ and $6\frac{1}{4}$, the three required numbers, as before, are $2\frac{1}{2}$, $6\frac{1}{4}$ and $15\frac{5}{8}$.

Note.—In dividing by the square of the ratio $+1$, when the first term is greater than the ratio the quotient will be greater than y^2: in this case take the nearest square root to the quotient; then multiply the remainder by the ratio $+1$, and add that product to the former remainder, and their sum will equal x^2. Thus:

5. Given the sum of three numbers in geometrical progression equal 65, and the sum of their squares equal 2275; to find those three numbers; the ratio being 3.

First $r^2+1=10$, and $2275 \div 10=227+5$; but 227 is not a square number; then, by extracting the nearest root, we have $15^2=225$, and $227-225=2$; then $2\times 10=20+$ the former remainder $5=25$, and $\sqrt{25}=5=x$; and thus having found the two first numbers to be 5 and 15, the three required numbers are 5, 15 and 45.

Or thus, when no ratio is given: $2275 \div 3=758\frac{1}{3}=m^2+\frac{1}{3}d^2$; then the number, the square of which is less than $758\frac{1}{3}$, and leaves a remainder equal to one-third of a square number, is $25^2=625$; then $758\frac{1}{3}-625=133\frac{1}{3}=\frac{1}{3}d^2$; then $133\frac{1}{3}\times 3=400=d^2$, and $\sqrt{400}=20=d$; therefore $25-20=5=x$, and $25+20=45=z$, and $\sqrt{xz}=15=y$; and the three required numbers, as before, are 5, 15 and 45.

6. Given the sum of the squares of the two extremes of three numbers in geometrical progression equal 2313; to find those three numbers.

Let x, y and z, and other symbols, be as before; then the $2313 \div 2=1156\frac{1}{2}=m^2+d^2$; then the number, the square of which is less than $1156\frac{1}{2}$, and leaves q equal to a square number, is $25\frac{1}{2}$, and $25\frac{1}{2}^2$ or $m^2=650\frac{1}{4}$; then $1156\frac{1}{2}-650\frac{1}{4}=506\frac{1}{4}=d^2$, and $\sqrt{506\frac{1}{4}}=22\frac{1}{2}=d$; then $25\frac{1}{2}-22\frac{1}{2}=3=x$, and $25\frac{1}{2}+22\frac{1}{2}=48=y$; thus the two extremes are 3 and 48; then $xz=144=y^2$, and $\sqrt{144}=12=y$; and the three required numbers are 3, 12 and 48.

7. Given the square of the middle of three numbers in geometrical progression equal 324; to find those numbers.

First $\sqrt{324}=18=y$, also $y^2=xz$; we then have given the product of two numbers equal 324, to find those numbers, as per case second; when by a few trials we find that the number, the square of which is greater than 324, and leaves a square number, is $30^2=900$; then $900-324=576=d^2$, and $\sqrt{576}=24=d$; therefore $30-24=6=x$, and $30+24=54=y$; and the three required numbers are 6, 18 and 54.

NOTE.—In this case, by having the sum of the three numbers given, the work will be done more easily. Thus:

Given the sum of three numbers in geometrical progression equal 93, and the square of the middle number equal 225; to find those three numbers.

First $\sqrt{225}=15=y$; and as the sum is given equal 93, then $93-15=78$, the sum of the two extreme numbers; we then have given the sum of two numbers equal 78, and their product equal 225, to find those numbers; then $78\div2=39=m$, and $m^2=1521$; then $m^2-y^2=1296=d^2$, and $\sqrt{1296}=36=d$; therefore $39-36=3=x$, and $39+36=75=z$; and the three required numbers are 3, 15 and 75.

CASE XIX.

The continued product of three numbers in geometrical progression given; to find those numbers.

RULE.

Propositions of this nature, without further data, are in some measure unlimited. The continued product of any three numbers in geometrical progression is always equal to the cube of the middle number, and any number by which that is divisible will be a ratio, and will give the same product. Thus: Given the continued product of three numbers in geometrical progression equal 1728; to find those numbers. Then $\sqrt[3]{1728}=12$, the middle number, and 12 is divisible by 2, by 3, by 4, and by 6; then if we take any one of these divisors, and produce from it 12 for the middle number, we shall have $2\times6=12$, $3\times4=12$, $4\times3=12$. In the first, we shall have this progress: $2\times12\times72=1728$; in the second, we shall have $3\times12\times48=1728$; in the third, we shall have $4\times12\times36=1728$; to which might be added 1; as $1\times12\times144=1728$; but if the sum of the three numbers be given, the problem then is limited.

EXAMPLES.

1. Given the sum of three numbers in geometrical progression equal 63, and their continued product equal 1728; to find those three numbers.

First $\sqrt{1728}=12=y$, and $y^2=144=xz$, and $63-12=51$; we then have these data, viz. the sum of two numbers equal 51, and their product equal 144, to find those two numbers; then $51\div 2=25\frac{1}{2}=m$, and $m^2=650\frac{1}{4}$; then $650\frac{1}{4}-144=506\frac{1}{4}$, and $\sqrt{506\frac{1}{4}}=22\frac{1}{2}=d$; therefore $25\frac{1}{2}-22\frac{1}{2}=3=x$, and $25\frac{1}{2}+22\frac{1}{2}=48=z$; and the three required numbers are 3, 12 and 48.

2. Given the sum of three numbers in geometrical progression equal 172, and their continued product equal 13824; to find those three numbers.

First $\sqrt[3]{13824}=24$, and $172-24=148$; therefore $24=y$, and $y^2=576$; we then have these data, viz. the sum of two numbers equal 148, and their product equal 576, to find those two numbers; then $148\div 2=74=m$, and $m^2=5476$; then $5476-576=4900$, and $\sqrt{4900}=70=d$; therefore 24 is the mean proportional, and $74-70=4=x$, and $74+70=144=z$; and having before found $y=24$, the three required numbers are 4, 24 and 144.

3. Given the sum of three numbers in geometrical progression equal 39, and the product of the two extremes equal 81; to find those three numbers.

First $\sqrt{81}=9=y$, and $39-9=30=x+z$; we then have the sum of two numbers equal 30, and their product equal 81, to find those two numbers; then $30\div 2=15=m$, and $m^2=225$; then $225-81=144=d^2$, and $\sqrt{144}=12=d$; therefore $15-12=3$, and $15+12=27$; and having found $y=9$, the three required numbers are 3, 9 and 27.

4. Given the sum of three numbers in geometrical progression equal 124, and the product of the two first equal 80; to find those three numbers.

On trial it will be found that $9^2=81$ will be greater than 80, and leave a square number equal 1; then $9-1=8$, and $9+1=10$, and $10\times 8=80$; but this would not give three numbers in geometrical progression equal 124; then by further trial it will be found that 12 is the next number of which the square will leave a difference between it and 80 equal a square number; then put $12=m$, and $m^2=144$; then $144-80=64=d^2$, and $\sqrt{64}=8=d$; therefore $12-8=4=x$, and $12+8=20=y$; and having found the two first numbers, the three required numbers will of course be 4, 20 and 100, whose sum is 124.

CASE XX.

The first term of four numbers in geometrical progression, their sum, and their continued product, being given; to find those numbers.

RULE.

The product, divided by the square of the first term, gives the square of the last term; which, divided by the first, gives the cube of the ratio; then by dividing the fourth term by the ratio, we get the third term, which of course will give all the others. Or thus: Let w, x, y and z equal the four required numbers, p their product, and r their ratio; then will $p \div w^2 = z^2$, and $z \div w = r^3$; then $z \div r = y$, and $y \div r = x$, &c.

EXAMPLES.

1. Given the first of four numbers in geometrical prosion equal 3, and their continued product equal 59049; to find the ratio and the four numbers.

First $w^2=9$, and $59049 \div 9 = 6561 = z^2$; then $\sqrt{6581} = 81$, and $81 \div w = 27 = r^3 = 3$; then $81 \div 3 = 27 = y$, and $27 \div 3 = 9 = x$; and the four required numbers are 8, 9, 27 and 81.

In this problem the first term and the ratio being the same, the second term is equal to the square of the first; $x=w^2$, $y=w^3$, and $z=w^4$.

2. Given the first of four numbers in geometrical progression equal 7, and their continued product equal 37515627; to find the ratio and those four numbers.

First $w^2=49$, and $37515627 \div 49$ or $p \div w^2 = 765625 = z^2$, and $\sqrt{765625} = 875 = z$; then $z \div w = 125 = r^3$, and $\sqrt[3]{125} = 5 = r$; then $z \div r = 175 = y$, and $y \div r = 35 = x$; and the four required numbers are 7, 35, 175 and 875.

Or these problems may be solved without any data except the product. Thus:

Let w, x, y and z equal the four required numbers, p the product, m the mean proportional between the two means, r the ratio, and $2d$ the difference of the two means; then will $\sqrt{p} = xy$ also wz; we then have the product of the two means, also of the two extremes, to find those numbers (see case second); which when found will of course determine the ratio by which the four required numbers can be found.

EXAMPLES.

3. Given the continued product of four numbers in geometrical progression equal 11664; to find the ratio and those four numbers.

First $\sqrt{11664}=108=xy$; then the number, the square of which is greater than $\sqrt{p}$, and will leave a remainder equal to a square number, is $12^2=144$; then put $12=m$, and $m^2-\sqrt{p}=36=d^2$, and $\sqrt{36}=6=d$; therefore $12-6=6=x$, and $12+6=18=y$; and having thus found x and y, and the ratio, the four required numbers are 2, 6, 18 and 54.

4. Given the continued product of four numbers in geometrical progression equal 2985984; to find the ratio and those four numbers.

First $\sqrt{p}=1728=xy$; then on trial it will be found that 48^2 is greater than 1728, and will leave a difference equal to a square number; then let $48=m$, and $m^2=2304$; then $m^2-\sqrt{p}=576=d^2$, and $\sqrt{576}=24=d$; therefore $48-24=24=x$, and $48+24=72=y$; and thus having found x and y, and the ratio, the four required numbers will of course be 8, 24, 72 and 216.

5. Given the product of the two extremes of four numbers in geometrical progression equal 243; to find the ratio and those four numbers.

First by a few trials we shall find $18^2=324-243=81$, which is a square number; then $\sqrt{81}=9$, and $18-9=9$, and $18+9=27$; but these cannot be the two extremes; then take these for the two means, and the ratio equal 3, and we find the two extremes thus, $9\div 3=3=w$, and $27\times 3=81=z$; and thus we have the four required numbers 3, 9, 27 and 81. Or by further trials we shall find 42^2 will leave a difference equal to a square number; then let $42=m$, and $m^2-p=1521=d^2$, and $\sqrt{1521}=39=d$; therefore $42-39=3=w$, and $42+39=81=z$; and thus having found the two extremes to be 3 and 81, then $z\div w=27=r^3$, and $\sqrt[3]{27}=3=r$; and the four required numbers are, as before, 3, 9, 27 and 81.

Or by having the ratio given, thus, $p\div r^2=x^4$, these problems may be solved thus:

6. Given the ratio of four numbers in geometrical progression equal 3, and their continued product equal 11664; to find those four numbers.

First $r^2=9$, and $p\div r^2=1296=x^4$; then $\sqrt[4]{1296}=6=x$; and thus having the ratio and the second term, the four required numbers are 2, 6, 18 and 54.

7. Given the ratio of four numbers in geometrical progression equal 4, and their continued product equal 331776; to find those four numbers.

First $r^2=16$; then $p\div r^2=20736=x^4$, and $\sqrt[4]{20736}=12=x$; thus having found x, and the ratio being given, the four required numbers are 3, 12, 48 and 192.

CASE XXI.

The ratio of four numbers in geometrical progression and the sum of their squares being given; to find those numbers.

RULE.

Divide the sum of the squares by the square of the ratio $+1$, when the quotient will be equal to the square of the third term more the square of the first; then the product of the first and third is equal to the square of the second; and thus having found the three first terms, the fourth is given.

Or thus: Let w, x, y and z, be the four required numbers, r the ratio, and s^2 the sum of their squares; then $s^2\div r^2+1=y^2+w^2$, and $wy=x^2$.

EXAMPLES.

1. Given the ratio of four numbers in geometrical progression equal 2, and the sum of their squares equal 765; to find those four numbers.

First $r^2+1=5$, and $s^2\div(r+1)^2=153=y^2+w^2$; then $\sqrt{153}=12=y+9=w^2$, and $\sqrt{9}=3=w$; then $w\times y=36=x^2$, and $\sqrt{36}=6=x$; and having found the three first to be 3, 6 and 12, and the ratio to be 2, the four required numbers are 3, 6, 12 and 24.

2. Given the ratio of four numbers in geometrical progression equal 3, and the sum of their squares equal 13120; to find those four numbers.

First $r^2+1=10$, and $s^2\div10=1312=y^2+w^2$; then the $\sqrt{1312}=36=y+16=w^2$, and $\sqrt{16}=4=w$; then $w\times y=144=x^2$, and $\sqrt{144}=12=x$; and having found the three first numbers equal 4, 12 and 36, the four required numbers are of course 4, 12, 36 and 108.

Problems of this kind may also be solved without the ratio, by approximation; thus, extract the square root of the sum of the squares, or take such a root as will leave a remainder that is a square number; then the number that will divide this, and produce a quotient that is a square number,

will be (the square of the ratio less one); the first root will then equal the sum of the two extremes, and the remainder thus divided (or the quotient thus produced) will be equal to the square of the second term. Or thus: Let w, x, y and z equal the four numbers, s^2 the sum of their squares, and r the ratio; then will $\sqrt{s^2}=w+z$, and the remainder, divided by $(r-1)^2=x$.

3. Given the sum of the squares of four numbers in geometrical progression equal 340; to find those numbers.

First the square root nearest to 340 is 18, and $18^2=324$, and $s^2-18^2=16$, which is a square number; then $16\div1=16$, and $1=(r-1)^2$, and $\sqrt{16}=4=x$; and thus having found $x=4$, and $r=2$, the four required numbers are 2, 4, 8 and 16.

4. Given the sum of the squares of four numbers in geometrical progression equal 65104; to find those numbers.

First $\sqrt{s^2}$, to leave a remainder that is a square number, is $252=w+z+1600$; then the only number that will divide 1600, and give a quotient that is also a square number, is 16; then $1600\div16=100=x^2$, and $\sqrt{100}=10=x$; thus having found $x=10$, and $r-1=\sqrt{16}$, or the ratio equal 5, we have of course the four required numbers, 2, 10, 50 and 250.

CASE XXII.

The ratio of five numbers in geometrical progression and their continued product being given; to find those numbers.

RULE.

Let v, w, x, y and z equal the five required numbers, p their product, and r their ratio; then will $p\div r^5=w^5$, and $w^5\div r^5=v^5$.

EXAMPLES.

1. Given the ratio of five numbers in geometrical progression equal 2, and their continued product equal 32768; to find those five numbers.

First $r^5=32$, and $p\div r^5=1024=w^5$; then $\sqrt[5]{1024}=4=w$, and $w^5\div r^5=32=v^5$, and $\sqrt[5]{32}=2=v$; thus having found $v=2$, and $w=4$, the five required numbers are 2, 4, 8, 16 and 32.

2. Given the ratio of five numbers in geometrical progression equal 3, and their continued product equal 1889568; to find those five numbers.

First $r^5=243$, and $p\div r^5=7776=w^5$; then $\sqrt[5]{7776}=$

$6=w$, and $w^5\div r^5=32=v^5$, and $\sqrt[5]{32}=2=v$; thus having found $v=2$, and $w=6$, the five required numbers are 2, 6, 18, 54 and 162.

Or problems of this kind, without the ratio given, may be solved thus, $p=x^5$; and having found x, then $p\div x^2=2y^2$.

3. Given the continued product of five numbers in geometrical progression equal 32768; to find those numbers.

First $\sqrt[5]{32768}=8=x$, and $x^2=64$; then $p\div x^2=512=2y^2$; then $512\div 2=256=y^2$, and $\sqrt{256}=16=y$; thus having found $x=8$, $y=16$, and the ratio equal 2, the five required numbers are 2, 4, 8, 16 and 32.

4. Given the continued product of five numbers in geometrical progression equal 1889568; to find those numbers.

First $\sqrt[5]{p}=18=x$, and $x^2=324$; then $p\div x^2=5832=2y^2$, and $5832\div 2=2916$, and $\sqrt{2916}=54=y$; thus having found $x=18$, $y=54$, and the ratio equal 3, the five required numbers are 2, 6, 18, 54 and 162.

5. Given the continued product of five numbers in geometrical progression equal 254803968; to find those numbers.

First $\sqrt[5]{p}=48=x$, and $x^2=2404$; then $p\div x^2=110592=vy^2$; then to find v, a few trials will show that $110592\div 3=36864$, and $\sqrt{36864}=192=y^2$; and thus having found $x=48$, $y=192$, and the ratio equal 4, we have the five required numbers 3, 12, 48, 192 and 768.

CASE XXIII.

The ratio of five numbers in geometrical progression and the sum of their squares being given; to find those numbers.

RULE.

$$s^2\div r^2+1=y^2+w^2+v^2.$$

EXAMPLES.

1. Given the ratio of five numbers in geometrical progression equal 2, and the sum of their squares equal 1364; to find those five numbers.

First $r^2+1=5$, and $1364\div 5=272+4$; then $\sqrt{4}=2=v$, and the number, the square of which is less than 272, and leaves a difference equal to a square number, is 16; then $16^2=256$, and $272-256=16=w^2$, and $\sqrt{16}=4=w$; and thus having found $v=2$, $w=4$, and $y=16$, we have the five required numbers 2, 4, 8, 16 and 32.

2. Given the ratio of five numbers in geometrical progression equal 3, and the sum of their squares equal 29524; to find those five numbers.

First $r^2+1=10$, and $29524\div10=2952+4$; then $\sqrt{4}=2=v$, and the number, the square of which is less than 2952, and leaves a difference equal to a square number, is 54; then let $y=54$, and $54^2=2916$; then $2952-2916=36=w^2$. and $\sqrt{36}=6=w$; thus having found $v=2$, $w=6$, and $y=54$, the five required numbers are 2, 6, 18, 54 and 162.

3. Given the ratio of five numbers in geometrical progression equal 4, and the sum of their squares equal 629145; to find those five numbers.

First $r^2+1=17$, and $s^2\div r^2+1=37008+9$; then $\sqrt{9}=3=v$, and the number, the square of which is less than 37008, and leaves a difference equal to a square number, is 192; then put $192=y$, and $192^2=36864$; then $37008-36864=144=w^2$, and $\sqrt{144}=12=w$; thus having found $v=3$, $w=12$, and $y=192$, the five required numbers are 3, 12, 48, 192 and 768.

NOTE.—It will be observed, that, having the ratio given, we have the solution, after finding v, without further investigation. I have, however, pursued the theory, to prove its correctness.

CASE XXIV.

The sum of three numbers in harmonical proportion and their continued product being given; to find those three numbers.

RULE.

The continued product of any three numbers in harmonical proportion is equal to the cube of the middle term, more the cube of the difference between the middle term and the greater extreme; therefore find the nearest cube that is less than the product, and will leave a difference equal to a cube, and the solution is done; for one-half the difference between the mean and the greater extreme is the difference between the mean and the less extreme.

EXAMPLES.

1. Given the sum of three numbers in harmonical proportion equal 13, and their continued product equal 72; to find those three numbers.

Let x, m and z equal the three numbers, and d equal the

difference between the mean and the greater extreme; then the nearest cube to 72, but less, is $4^3=64=m^3$, and $72-64=8=d^3$, and $\sqrt[3]{8}=2=d$; therefore $4+2=6=z$, and $2\div2=1$, and $4-1=3=x$; and the three required numbers are 3, 4 and 6.

2. Given the sum of three numbers in harmonical proportion equal 26, and their continued product equal 576; to find those numbers.

Let x, m and z equal the three numbers, and d the difference between the mean and greater extreme, as before; then the nearest cube to 576, but less, is $8^3=512$; then $576-512=64=d^3$, and $\sqrt[3]{64}=4=d$; therefore $8=m$, $8+4=12=z$, and $8-2=6=x$; and the three required numbers are 6, 8 and 12.

3. Given the sum of three numbers in harmonical proportion equal $34\frac{2}{3}$, and their continued product equal $1365\frac{1}{3}$; to find those numbers.

Let x, m, z and d equal as before; then it will readily be observed that the nearest cube of that fraction, which is less than $1365\frac{1}{3}$, is $10\frac{2}{3}=1213\frac{1}{27}$; then $1365\frac{1}{3}-1213\frac{1}{27}=151\frac{19}{27}$ and $\sqrt[3]{151\frac{19}{27}}=5\frac{1}{3}=d$; therefore $10\frac{2}{3}=m$, $10\frac{2}{3}+5\frac{1}{3}=16=z$, and $10\frac{2}{3}-2\frac{2}{3}=8=x$; and the three required numbers are 8, $10\frac{2}{3}$ and 16.

CASE XXV.

The sum of three numbers in harmonical proportion and the sum of their squares being given; to find those three numbers.

RULE.

The sum of the squares of any three numbers in harmonical proportion is equal to three times the square of the mean or middle number, more three times the square of the difference between the mean and the greater extreme, more the square of the difference between the mean and the less extreme; therefore, by dividing by 3, we have $m^2+d^2+\frac{1}{3}\,\square\,\frac{1}{2}d$.

EXAMPLES.

1. Given the sum of three numbers in harmonical proportion equal 13, and the sum of their squares equal 61; to find those three numbers.

Let x, m and z equal the three numbers, and d the difference between m and z; then $61\div3=20\frac{1}{3}=m^2+d^2+\frac{1}{3}\,\square$

$\frac{1}{2}d$; then the nearest square number that is less than $20\frac{1}{3}$ is $4^2=16$; then put $4=m$, and $20\frac{1}{3}-16=4\frac{2}{3}=d^2+\frac{1}{3}\square\frac{1}{2}d$; then $\sqrt{4\frac{1}{3}}=2=d^2+\frac{1}{3}$, and $\frac{1}{3}\times3=1=\square\frac{1}{2}d$; therefore $4=m$, $4-1=3=x$, and $4+2=6=z$; and the three required numbers are 3, 4 and 6.

2. Given the sum of three numbers in harmonical proportion equal 26, and the sum of their squares equal 244; to find those three numbers.

First $244\div3=81\frac{1}{3}=m^2+d^2+\frac{1}{3}\square\frac{1}{2}d$; then the square that is less than $81\frac{1}{3}$ is $8^2=64$; then put $8=m$, and $81\frac{1}{3}-64=17\frac{1}{3}=d^2+\frac{1}{3}\square\frac{1}{2}d$; then $\sqrt{17\frac{1}{3}}=4=d+1\frac{1}{3}$, and $1\frac{1}{3}\times3=\square\frac{1}{2}d$, and $\sqrt{4}=2=\square\frac{1}{2}d$; therefore $8=m$, $8-2=6=x$, and $8+4=12=z$; and the three required numbers are, of course, 6, 8 and 12.

END OF THE FIRST NUMBER.